Based on the New Textbook

NAVNEET PHYSICS Digest

PART 1 (Chapters 1 to 6)

STANDARD XII

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PREFACE

Dear Friends,

It gives us great pleasure to present you this first edition of Navneet Physics Digest (Part I) prepared according to the new Textbook and the latest question paper pattern for Standard XII.

We understand that the Standard XII is very crucial in a student’s career. As always, Navneet Digests will help you to triumph.

This Physics Digest contains model answers and solutions to all the questions and problems given in the Board’s Textbook. There are also additional graded and varied questions with answers and solved problems so as to cover every concept in the Textbook. Besides these, all the stimulating in-text questions/informations under ‘Can you tell?’, ‘Do you know?’, ‘Use your brain power’, etc. are included (with answers wherever necessary). In short, this Physics Digest contains a lucid exposition of the syllabus in simple and clear language.

Wherever necessary, the answers have been illustrated with neat, accurate, easily reproducible diagrams, labelled as per the evaluation guidelines. Each diagram is given in an attractive two-colour style drawing your focus to its important feature(s).

Sufficient number of Multiple Choice Questions (MCQs) have been given which will be useful for Std. XII Board examination as well as other competitive examinations. A large number of well-graded Problems for Practice with answers, Weblinks, Formulae at a Glance and Memory Map are also given at the end of each chapter. You should use the Formulae at a Glance before attempting numerical problems and multiple choice questions. Memory Map gives you the overview of a chapter. You should use this for a systematic approach to your study as well as for effective revision.

The Internet my friend section include few links to authentic online study material, lectures, demonstrations and simulations. Follow these WWW links for a better understanding of the subject and additional knowledge that will serve you in good stead for any entrance examination.

We have taken utmost care to see that this Digest proves to be very useful to the students as well as the teachers. Suggestions for improvement of the Digest are always welcome and will be gratefully acknowledged and appreciated.

We hope this Digest with all its important features will help you to secure a high percentage of marks in the coming examination. We wish you all the best.

– The Publishers
## CONTENTS

<table>
<thead>
<tr>
<th>Chapter No.</th>
<th>Title</th>
<th>Page No.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Rotational Dynamics</td>
<td>... 5</td>
</tr>
<tr>
<td>2.</td>
<td>Mechanical Properties of Fluids</td>
<td>... 92</td>
</tr>
<tr>
<td>3.</td>
<td>Kinetic Theory of Gases and Radiation</td>
<td>... 149</td>
</tr>
<tr>
<td>4.</td>
<td>Thermodynamics</td>
<td>... 199</td>
</tr>
<tr>
<td>5.</td>
<td>Oscillations</td>
<td>... 234</td>
</tr>
<tr>
<td>6.</td>
<td>Superposition of Waves</td>
<td>... 280</td>
</tr>
</tbody>
</table>

Chapter No. 7 to 16 are included in Navneet Physics Digest : Part 2, published separately

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**NAVNEET — SIXTY DEVOTED YEARS IN THE SERVICE OF EDUCATION!**

Founded in 1959, **Navneet Education Limited** is now a sixty-year young organization. During this journey, Navneet has marked many significant milestones from publishing to stationery, the domestic to overseas market, and further now from Print Products to e-Learning. We joyously celebrate this Diamond Jubilee as a complete service provider in the field of education.

As often said: Quality is never an accident; it is always the result of intelligent effort. **The Founders of Navneet** took great effort to deliver quality. We continue to cherish their devotion and try our best to adhere to their norms.

The discerning students, parents, teachers, principals, educational institutions and booksellers have been our great source of strength. Their vision and confidence in quality products has helped us a lot. We look forward to reaching greater heights with their support and good wishes.

Our gratitude to all our well-wishers and workforce, who have helped us in our long journey and making NAVNEET a synonym for quality – par excellence.

**Regards,**

**Navneet Education Limited**
Q. 1. What is circular motion? (1 mark)
Ans. The motion of a particle along a complete circle or a part of it is called circular motion.

Q. 2. What is radius vector in circular motion? (1 mark)
Ans. For a particle performing circular motion, its position vector with respect to the centre of the circle is called the radius vector.

[Note: The radius vector has a constant magnitude,
equal to the radius of the circle. However, its direction changes as the position of the particle changes along the circumference.

Q. 3. What is the difference between rotation and revolution? (1 mark)

Ans. There is no physical difference between them. It is just a question of usage. Circular motion of a body about an axis passing through the body is called rotation. Circular motion of a body around an axis outside the body is called revolution.

Q. 4. State the characteristics of circular motion. (2 marks)

Ans.
(1) It is an accelerated motion: As the direction of velocity changes at every instant, it is an accelerated motion.
(2) It is a periodic motion: During the motion, the particle repeats its path along the same trajectory. Thus, the motion is periodic.

Q. 5. Explain angular displacement in circular motion. (2 marks)

Ans. The change in the angular position of a particle performing circular motion with respect to a reference line in the plane of motion of the particle and passing through the centre of the circle is called the angular displacement.

Infinitesimal angular displacement $\delta \theta$ in an infinitesimal time interval $\delta t \to 0$, is given a direction perpendicular to the plane of revolution by the right hand thumb rule.

Q. 6. Explain angular velocity. State the right hand thumb rule for the direction of angular velocity. (2 marks)

Ans. Angular velocity: The time rate of angular displacement of a particle performing circular motion is called the angular velocity.

(i) If the particle has an angular displacement $\delta \theta$ in a short time interval $\delta t$, its angular velocity

$$\vec{\omega} = \lim_{\delta t \to 0} \frac{\delta \theta}{\delta t} = \frac{d\theta}{dt}$$

(ii) $\vec{\omega}$ is a vector along the axis of rotation, in the direction of $d\theta$, given by the right hand thumb rule.

Right hand thumb rule: If the fingers of the right hand are curled in the sense of revolution of the particle (Fig. 1.2), then the outstretched thumb gives the direction of the angular displacement.

Fig. 1.2: Directions of angular velocity

[Note: Angular speed, $\omega = |\vec{\omega}| = \frac{d\theta}{dt}$ is also called angular frequency.]

Q. 7. Explain the linear velocity of a particle performing circular motion. OR Derive the relation between the linear velocity and the angular velocity of a particle performing circular motion. (2 marks)

Ans. Consider a particle performing circular motion in an anticlockwise sense, along a circle of radius $r$. In a very small time interval $\delta t$, the particle moves from point A to point B through a distance $\delta s$ and its angular position changes by $\delta \theta$. As the particle moves in its circular path, its angular position changes, say from $\theta_1$ at time $t$ to $\theta_2$ at a short time $\delta t$ later, Fig. 1.1. In the interval $\delta t$, the position vector $\vec{r}$ sweeps out an angle $\delta \theta = \theta_2 - \theta_1$. $\delta \theta$ is the magnitude of the change in the angular position of the particle.
\[ \delta \theta = \frac{\text{arc } AB}{\text{radius}} = \frac{\delta s}{r} \]

As \( \delta t \to 0 \), B will be very close to A and displacement \( \overrightarrow{AB} = \delta \overrightarrow{s} \) will be a straight line perpendicular to radius vector \( \overrightarrow{OA} = \overrightarrow{r} \).

By the right hand rule of cross product,
\[ \lim_{\delta t \to 0} \frac{\delta \overrightarrow{s}}{\delta t} = \left( \lim_{\delta t \to 0} \frac{\delta \overrightarrow{\theta}}{\delta t} \right) \times \overrightarrow{r} \]
\[ \therefore \frac{d\overrightarrow{s}}{dt} = \frac{d\overrightarrow{\theta}}{dt} \times \overrightarrow{r} \quad \text{... (1)} \]

![Fig. 1.3: \( \delta \overrightarrow{s} = \delta \overrightarrow{\theta} \times \overrightarrow{r} \)](image)

The linear velocity \( \overrightarrow{v} \) of the particle is the time rate of displacement and its angular velocity \( \overrightarrow{\omega} \) is the time rate of angular displacement.
\[ \therefore \overrightarrow{v} = \frac{d\overrightarrow{s}}{dt} \quad \text{and} \quad \overrightarrow{\omega} = \frac{d\overrightarrow{\theta}}{dt} \]

Since \( d\overrightarrow{s} \) is tangential, the instantaneous linear velocity \( \overrightarrow{v} \) of a particle performing circular motion is along the tangent to the path, in the sense of motion of the particle. \( \overrightarrow{v} \), \( \overrightarrow{\omega} \) and \( \overrightarrow{r} \) are mutually perpendicular, so that in magnitude, \( v = r \omega \).

Q. 8. State the relation between the linear velocity and the angular velocity of a particle in circular motion. \hspace{1cm} (1 mark)
Ans. Linear velocity, \( \overrightarrow{v} = \overrightarrow{\omega} \times \overrightarrow{r} \) where \( \overrightarrow{\omega} \) is the angular velocity and \( \overrightarrow{r} \) is the radius vector.
At every instant, \( \overrightarrow{v} \), \( \overrightarrow{\omega} \) and \( \overrightarrow{r} \) are mutually perpendicular, so that in magnitude \( v = r \omega \).

Q. 9. Define uniform circular motion (UCM). \hspace{1cm} (1 mark)
Ans. A particle is said to perform uniform circular motion if it moves in a circle or a circular arc at constant linear speed or constant angular velocity.

Q. 10. A stone tied to a string is rotated in a horizontal circle (nearly). If the string suddenly breaks, in which direction will the stone fly off? \hspace{1cm} (1 mark)
Ans. In a circular motion, the instantaneous velocity \( \overrightarrow{v} \) is always tangential, in the sense of the motion. Hence, an inertial observer will see the stone fly off tangentially, in the direction of \( \overrightarrow{v} \) at the instant the string breaks.

Q. 11. What is the angular speed of a particle moving in a circle of radius \( r \) centimetres with a constant speed of \( v \) cm/s? \hspace{1cm} (1 mark)
Ans. Angular speed, \( \omega = \frac{v \text{ cm/s}}{r \text{ cm}} = \frac{v}{r} \text{ rad/s} \).

Q. 12. Define the period and frequency of revolution of a particle performing uniform circular motion (UCM) \hspace{1cm} (2 marks) and state expressions for them. \hspace{1cm} (2 marks) Also state their SI units. \hspace{1cm} (1 mark)
Ans.

(1) Period of revolution: The time taken by a particle performing UCM to complete one revolution is called the period of revolution or the period \( (T) \) of UCM.
\[ T = \frac{2\pi r}{v} = \frac{2\pi}{\omega} \]
where \( v \) and \( \omega \) are the linear and angular speeds, respectively.

SI unit : the second \( (s) \)
Dimensions : \( [M^0L^0T^1] \).

(2) Frequency of revolution: The number of revolutions per unit time made by a particle in UCM is called the frequency of revolution \( (f) \).

The particle completes 1 revolution in periodic time \( T \). Therefore, it completes \( 1/T \) revolutions per unit time.
\[ \therefore \text{Frequency } f = \frac{1}{T} = \frac{v}{2\pi r} = \frac{\omega}{2\pi} \]
SI unit : the hertz \( (Hz) \), 1 Hz = 1 s\(^{-1}\)
Dimensions : \( [M^0L^0T^{-1}] \).

Q. 13. If the angular speed of a particle in UCM is \( 20 \pi \) rad/s, what is the period of UCM of the particle? \hspace{1cm} (1 mark)
Ans. The period of UCM of the particle,
\[ T = \frac{2\pi}{\omega} = \frac{2\pi}{20\pi} = 0.1 \text{ s} \]
Q. 14. Why is UCM called a periodic motion? (1 mark)

Ans. In a uniform motion, a particle covers equal distances in equal intervals of time. Any motion which repeats itself in equal intervals of time is called a periodic motion. In a uniform circular motion (UCM), the particle takes the same time to complete each revolution, a distance equal to the circumference of the circle. Therefore, it is a periodic motion.

Q. 15. Give one example of uniform circular motion. (1 mark)

Ans.

(1) Circular motion of every particle of the blades of a fan or the dryer drum of a washing machine when the fan or the drum is rotating with a constant angular speed.

(2) Motion of the hands of a clock.

(3) Motion of an Earth-satellite in a circular orbit.

Q. 16. What can you say about the angular speed of an hour hand as compared to that of the Earth's rotation about its axis? (1 mark)

Ans. The periods of rotation of an hour hand and the Earth are $T_h = 12 \text{ h}$ and $T_E = 24 \text{ h}$, respectively, so that their angular speeds are $\omega_h = \frac{2\pi}{12} \text{ rad/h}$ and $\omega_E = \frac{2\pi}{24} \text{ rad/h}$.

$\therefore \omega_h = 2\omega_E$

Q. 17. Explain the acceleration of a particle in UCM. State an expression for the acceleration. (3 marks)

Ans. A particle in uniform circular motion (UCM) moves in a circle or circular arc at constant linear speed $\vec{v}$. The instantaneous linear velocity $\vec{v}$ of the particle is along the tangent to the path in the sense of motion of the particle. Since $\vec{v}$ changes in direction, without change in its magnitude, there must be an acceleration that must be (i) perpendicular to $\vec{v}$ (ii) constant in magnitude (iii) at every instant directed radially inward, i.e., towards the centre of the circular path.

Such a radially inward acceleration is called a centripetal acceleration.

\[ a = \frac{d\vec{v}}{dt} = a_t \]

If $\omega$ is the constant angular velocity of the particle and $r$ is the radius of the circle,

\[ a_t = -\omega^2 r \]

where $\omega = |\omega|$ and the minus sign shows that the direction of $a_t$ is at every instant opposite to that of the radius vector $r$. In magnitude,

\[ a_t = \omega^2 r = \frac{v^2}{r} = \omega v \]

[Note: The word centripetal comes from Latin for ‘centre-seeking’.]

Q. 18. Draw a diagram showing the linear velocity, angular velocity and radial acceleration of a particle performing circular motion with radius $r$. (1 mark)

Ans.

Fig. 1.4: Directions of $\omega$, $\vec{v}$ and $a_t$ for a particle P revolving in the anticlockwise sense

Q. 19. If a particle in UCM has linear speed 2 m/s and angular speed 5 rad/s, what is the magnitude of the centripetal acceleration of the particle? (1 mark)

Ans. The magnitude of the centripetal acceleration of the particle is

\[ a_t = \omega v = (5)(2) = 10 \text{ m/s}^2 \]

Q. 20. State any two quantities that are uniform in UCM. (1 mark)

Ans. Linear speed and angular speed. (Also, kinetic energy, angular speed and angular momentum.)

Q. 21. State any two quantities that are nonuniform in UCM. (1 mark)

Ans. Velocity and acceleration are nonuniform in UCM. (Also, centripetal force.)

Q. 22. What is a nonuniform circular motion? (1 mark)

Ans. Consider a particle moving in a plane along a circular path of constant radius. If the particle is
speeding up or slowing down, its angular speed \( \omega \) and linear speed \( v \) both change with time. Then, the particle is said to be in a nonuniform circular motion.

Q. 23. Explain angular acceleration. (2 marks)

Ans. Angular acceleration: The time rate of change of angular velocity of a particle performing circular motion is called the angular acceleration.

(i) If \( \Delta \omega \) is the change in angular velocity in a short time interval \( \Delta t \), the angular acceleration

\[ \alpha = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt} \]

(ii) The direction of \( \alpha \) is the same as that of \( \frac{d\omega}{dt} \).

We consider the case where a change in \( \omega \) arises due to a change in its magnitude only. If the particle is speeding up, i.e., \( \omega \) is increasing with time, then \( \alpha \) is in the direction of \( \omega \). If the particle is slowing down, i.e., \( \omega \) is decreasing with time, then \( \alpha \) is directed opposite to \( \omega \).

![Fig. 1.5: Angular acceleration \( \alpha \) when \( \omega \) is (a) increasing (b) decreasing](image)

(iii) If the angular speed changes from \( \omega_1 \) to \( \omega_2 \) in time \( t \), the magnitude \( (\alpha) \) of the average angular acceleration is

\[ \alpha = \frac{\omega_2 - \omega_1}{t} \]

Q. 24. Explain the tangential acceleration of a particle in nonuniform circular motion. (3 marks)

Ans. Tangential acceleration: For a particle performing circular motion, the linear acceleration tangential to the path that produces a change in the linear speed of the particle is called the tangential acceleration.

Explanation:

(i) If a particle performing circular motion is speeding up or slowing down, its angular speed \( \omega \) and linear speed \( v \) both change with time.

![Fig. 1.6: Tangential acceleration when linear speed is (a) increasing (b) decreasing](image)

The tangential acceleration that produces a change only in the linear speed must be along \( \vec{v} \). Hence, it is called the tangential acceleration, \( \vec{a}_t \). In magnitude,

\[ a_t = \frac{dv}{dt} \]

(ii) If the linear speed \( v \) of the particle is increasing, \( \vec{a}_t \) is in the direction of \( \vec{v} \). If \( v \) is decreasing, \( \vec{a}_t \) is directed opposite to \( \vec{v} \) (Fig. 1.6).

Q. 25. Obtain the relation between the magnitudes of the linear (tangential) acceleration and angular acceleration in nonuniform circular motion. (2 marks)

Ans. Consider a particle moving along a circular path of constant radius \( r \). If the particle is speeding up or slowing down, its motion is nonuniform, and its angular speed \( \omega \) and linear speed \( v \) both change with time. At any instant, \( v, \omega \) and \( r \) are related by

\[ v = \omega r \]

The angular acceleration of the particle is

\[ \alpha = \frac{d\omega}{dt} \]

The tangential acceleration \( \vec{a}_t \) is the linear acceleration that produces a change in the linear speed of the particle and is tangent to the circle. In magnitude,

\[ a_t = \frac{dv}{dt} = \frac{d}{dt}(\omega r) = \left( \frac{d\omega}{dt} \right) r \quad (\because r \text{ is a constant}) \]

\[ \therefore a_t = \alpha r \]

This is the required relation.

Q. 26. Obtain an expression for the acceleration of a particle performing circular motion. Explain its two components. OR

For a particle performing uniform circular motion, \( \vec{v} = \omega \times \vec{r} \). Obtain an expression for the
linear acceleration of a particle performing non-uniform circular motion.

\[ \vec{a} = \omega \times \vec{v} \]

In circular motion, assuming \( \vec{v} = \omega \times \vec{r} \), obtain an expression for the resultant acceleration of a particle in terms of tangential and radial components.

(2 marks)

Ans. Consider a particle moving along a circular path of constant radius \( r \). If its motion is nonuniform, then its angular speed \( \omega \) and linear speed \( v \) both change with time.

Therefore, in general, the particle has both angular acceleration \( \vec{\alpha} = \frac{d\omega}{dt} \) and tangential acceleration \( \vec{a}_t \).

\( \vec{\alpha} \) has the direction of \( \dot{\omega} \), which is in the direction of \( \vec{a}_t \) if \( \omega \) is increasing and opposite to \( \vec{\omega} \) if \( \omega \) is decreasing.

At any instant, the linear velocity \( \vec{v} \), angular velocity \( \vec{\omega} \) and radius vector \( \vec{r} \) are related by

\[ \vec{v} = \vec{\omega} \times \vec{r} \]

... (1)

The linear acceleration of the particle is

\[ \vec{a} = \frac{d\vec{v}}{dt} \]

... (2)

\[ \vec{a} = \frac{d}{dt} (\vec{\omega} \times \vec{r}) = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt} \]

... (3)

\( \vec{\omega} \times \vec{r} \) is tangential to the circular path and is in the direction of \( \vec{v} \) if \( \vec{\omega} \) is in the direction of \( \vec{r} \), and it is opposite to \( \vec{v} \) if \( \vec{\omega} \) is opposite to \( \vec{r} \). Thus, \( \vec{\omega} \times \vec{r} \) is the tangential acceleration, \( \vec{a}_t \).

\[ \vec{a}_t = \vec{\omega} \times \vec{r} \]

... (4)

In magnitude, \( a_t = v \) since \( \vec{\omega} \) is perpendicular to \( \vec{r} \).

Also, \( \vec{\omega} \times \vec{v} \) is along the radius towards the centre of the circle, i.e., opposite to \( \vec{r} \), i.e., along \(-\vec{r}\); this acceleration is called the radial or centripetal acceleration \( \vec{a}_r \).

\[ \vec{a}_r = \vec{\omega} \times \vec{v} \]

... (5)

In magnitude, \( a_r = \omega v \) since \( \vec{\omega} \) is perpendicular to \( \vec{v} \).

: \[ \vec{a} = \vec{a}_t + \vec{a}_r \] ... (6)

This is the required expression.

Q. 27. What is the angle between linear acceleration and angular acceleration of a particle in nonuniform circular motion?

(1 mark)

Ans. In a nonuniform circular motion, the angular acceleration is an axial vector, perpendicular to the plane of the motion. The linear acceleration is in the plane of the motion. Hence, the angle between them is 90°.

Q. 28. What are the differences between a nonuniform circular motion and a uniform circular motion? (Two points of distinction) Give examples.

(3 marks)

Ans.

(i) Nonuniform circular motion:

(1) The angular and tangential accelerations are non-zero, so that linear and angular speeds both change with time.

\[ \vec{\alpha} = \frac{d\vec{\omega}}{dt} \quad \text{and} \quad \vec{a}_t = \frac{d\vec{v}}{dt} = \vec{\omega} \times \vec{r} \]

If a particle in nonuniform circular motion is speeding up, \( \vec{\omega} \) is in the direction of \( \vec{v} \) and \( \vec{a}_t \) is in the direction of \( \vec{v} \); if the particle is slowing down, \( \vec{\omega} \) is opposite to \( \vec{v} \) and \( \vec{a}_t \) is opposite to \( \vec{v} \).

(2) The net linear acceleration, being the resultant of the radial and tangential accelerations, is not radial. \( \vec{a} = \vec{a}_t + \vec{a}_r \)

(3) The magnitudes of the centripetal acceleration and the centripetal force are not constant.

(4) Example: Motion of the tip of a fan blade when the fan is speeding up or slowing down.

(ii) Uniform circular motion:

(1) The angular and tangential accelerations are zero, so that linear speed and angular velocity are constant.

(2) The net linear acceleration is radially inward, i.e., centripetal.

(3) The magnitudes of the centripetal acceleration and the centripetal force are also constant.

(4) Example: Motion of the tips of the hands of a mechanical clock.
Q. 29. Write the kinematical equations for circular motion in analogy with linear motion.

\( (\frac{1}{2} \text{ mark each}) \)

Ans. For circular motion of a particle with constant angular acceleration \( \dot{\alpha} \),

\[ \frac{\omega}{t} = \frac{\omega_0 + \omega}{2} \]

where \( \omega_0 \) and \( \omega \) are the initial and final angular speeds, \( t \) is the time, \( \omega_{av} \) the average angular speed and \( \theta_0 \) and \( \theta \) the initial and final angular positions of the particle.

Then, the angular kinematical equations for the circular motion are (in analogy with linear kinematical equations for constant linear acceleration)

\[ \omega = \omega_0 + \dot{\alpha} t \]

\[ \theta - \theta_0 = \omega_0 t + \frac{1}{2} \dot{\alpha} t^2 \]

\[ \omega^2 = \omega_0^2 + 2 \dot{\alpha} (\theta - \theta_0) \]

---

Solved Problems 1.2.1

Q. 30. Solve the following:

(1) Certain stars are believed to be rotating at about 1 rot/s. If such a star has a diameter of 40 km, what is the linear speed of a point on the equator of the star? (2 marks)

Solution:

Data: \( d = 40 \text{ km}, f = 1 \text{ rot/s} \)

\[ r = \frac{d}{2} = \frac{40}{2} = 20 \text{ km} = 2 \times 10^4 \text{ m} \]

Linear speed, \( v = \omega r = (2\pi f) r \)

\[ = (2 \times 3.142 \times 1)(2 \times 10^4) \]

\[ = 6.284 \times 2 \times 10^4 \]

\[ = 1.257 \times 10^5 \text{ m/s} \text{ (or 125.6 km/s)} \]

(2) A body of mass 100 grams is tied to one end of a string and revolved along a circular path in the horizontal plane. The radius of the circle is 50 cm.

If the body revolves with a constant angular speed of 20 rad/s, find the (i) period of revolution (ii) linear speed (iii) centripetal acceleration of the body. (3 marks)

Solution:

Data: \( m = 100 \text{ g} = 0.1 \text{ kg}, r = 50 \text{ cm} = 0.5 \text{ m}, \omega = 20 \text{ rad/s} \)

(i) The period of revolution of the body,

\[ T = \frac{2\pi}{\omega} = \frac{2 \times 3.142}{20} = 0.3142 \text{ s} \]

(ii) Linear speed, \( v = \omega r = 20 \times 0.5 = 10 \text{ m/s} \)

(iii) Centripetal acceleration,

\[ a_c = \omega^2 r = (20)^2 \times 0.5 = 200 \text{ m/s}^2 \]

(3) Calculate the angular speed of the Earth due to its spin (rotational motion). (2 marks)

Solution:

Data: \( T = 24 \text{ hours} = 24 \times 60 \times 60 \text{ s} \)

Angular speed, \( \omega = \frac{2\pi}{T} = \frac{2 \times 3.142}{24 \times 60 \times 60} = 3.142 \]

\[ \frac{43200}{2400} = 7.273 \times 10^{-5} \text{ rad/s} \]

The angular speed of the Earth due to its spin (rotational motion) is \( 7.273 \times 10^{-5} \text{ rad/s} \).

(4) Find the angular speed of rotation of the Earth so that bodies on the equator would feel no weight. [Radius of the Earth = 6400 km, \( g = 9.8 \text{ m/s}^2 \)] (2 marks)

Solution:

Data: Radius of the Earth = \( r = 6400 \text{ km} = 6.4 \times 10^6 \text{ m}, g = 9.8 \text{ m/s}^2 \)

As the Earth rotates, the bodies on the equator revolve in circles of radius \( r \).

1. ROTATIONAL DYNAMICS
These bodies would not feel any weight if their centripetal acceleration \((\omega^2 r)\) is equal to the acceleration due to gravity \((g)\).

\[
\therefore \omega^2 r = g
\]

The angular speed of the Earth’s rotation,

\[
\omega = \sqrt{\frac{g}{r}} = \sqrt{\frac{9.8}{6.4 \times 10^6}} = 1.237 \times 10^{-3} \text{ rad/s}
\]

(5) To simulate the acceleration of large rockets, astronauts are seated in a chamber and revolved in a circle of radius 9.8 m. What angular speed is required to generate a centripetal acceleration 8 times the acceleration due to gravity?

\([g = 9.8 \text{ m/s}^2]\) \hspace{1cm} (2 marks)

**Solution:**

**Data:** \(r = 9.8 \text{ m}, g = 9.8 \text{ m/s}^2\), \(a = 8g\)

Centripetal acceleration = \(\omega^2 r\)

\[
\therefore \omega^2 r = 8g
\]

\[
9.8 \omega^2 = 8(9.8)
\]

\[
\therefore \omega^2 = 8
\]

The required angular speed,

\[
\omega = \sqrt{8} = 2 \sqrt{2} = 2.828 \text{ rad/s}
\]

(6) The angular position of a rotating object is given by \(\theta(t) = (1.55t^2 - 7.75t + 2.87)\) rad, where \(t\) is measured in second. (i) When is the object momentarily at rest? (ii) What is the magnitude of its angular acceleration at that time? \hspace{1cm} (3 marks)

**Solution:**

\(\theta(t) = (1.55t^2 - 7.75t + 2.87)\) rad

The angular speed of the object as a function of time is

\[
\omega(t) = \frac{d\theta}{dt} = \frac{d}{dt}(1.55t^2 - 7.75t + 2.87)
\]

\[
= 2(1.55t) - 7.75
\]

\[
= (3.10t - 7.75) \text{ rad/s}
\]

(i) When the object is momentarily at rest,

\[
\omega = 3.10t - 7.75 = 0
\]

\[
\therefore t = \frac{7.75}{3.10} = 2.5 \text{ s}
\]

(ii) The magnitude of the angular acceleration is

\[
\alpha = \frac{d\omega}{dt} = \frac{d}{dt}(3.10t - 7.75)
\]

\[
= 3.10 \text{ rad/s}^2 \text{ (constant)}
\]

(7) A motor part at a distance of 1.5 m from the motor’s axis of rotation has a constant angular acceleration of 0.25 rad/s². Find the magnitude of its linear acceleration at the instant when its angular speed is 0.5 rad/s. \hspace{1cm} (3 marks)

**Solution:**

**Data:** \(r = 1.5 \text{ m}, \alpha = 0.25 \text{ rad/s}^2, \omega = 0.5 \text{ rad/s}\)

\(\alpha = \omega^2 r = (0.5)^2(1.5) = 0.25 \times 1.5 = 0.375 \text{ m/s}^2\)

The linear acceleration,

\[
a = \sqrt{a_r^2 + a_t^2} = \sqrt{(0.375)^2 + (0.375)^2}
\]

\[
= \sqrt{2(0.375)^2} = \sqrt{28.12 \times 10^{-2}}
\]

\[
= 0.5303 \text{ m/s}^2
\]

(8) A coin is placed on a stationary disc at a distance of 1 m from the disc’s centre. At time \(t = 0\) s, the disc begins to rotate with a constant angular acceleration of 2 rad/s² around a fixed vertical axis through its centre and perpendicular to its plane. Find the magnitude of the linear acceleration of the coin at \(t = 1.5\) s. Assume the coin does not slip. \hspace{1cm} (3 marks)

**Solution:**

**Data:** \(r = 1 \text{ m}, \alpha = 2 \text{ rad/s}^2, \omega_0 = 0, t = 1.5 \text{ s}\)

\(\alpha = \omega = \omega_0 + \omega t = 0 + (2)(1.5) = 3 \text{ rad/s}\)

Angular speed at \(t = 1.5 \text{ s},\)

\[
\omega = 3 \text{ rad/s}
\]

\(\therefore \alpha = \omega^2 r = (3)^2(1) = 9 \text{ m/s}^2\)

The required linear acceleration is,

\[
a = \sqrt{a_r^2 + a_t^2} = \sqrt{9^2 + 2^2} = \sqrt{85}
\]

\[
= 9.22 \text{ m/s}^2
\]

[OR \(v = u + at = 0 + (2)(1.5) = 3 \text{ m/s}\)]

\(\therefore \alpha = \frac{v^2}{r} = \frac{3^2}{1} = 9 \text{ m/s}^2\]

(9) A railway locomotive enters a stretch of track, which is in the form of a circular arc of radius 280 m, at 10 m/s and with its speed increasing uniformly. Ten seconds into the stretch its speed is 14 m/s and at 18 s its speed is 19 m/s. Find (i) the magnitude of the locomotive’s linear acceleration when its speed is 14 m/s (ii) the direction of this acceleration at that point with respect to the locomotive’s radial acceleration (iii) the angular acceleration of the locomotive. \hspace{1cm} (4 marks)
Solution:

Data: \( r = 280 \text{ m}, v_1 = 10 \text{ m/s at } t_1 = 0, \)
\( v_2 = 14 \text{ m/s at } t_2 = 10 \text{ s}, v_3 = 19 \text{ m/s at } t_3 = 18 \text{ s} \)

(i) At \( t = t_2 \), the radial acceleration is
\[
a_r = \frac{v_2^2}{r} = \frac{(14)^2}{280} = 0.7 \text{ m/s}^2
\]

Since the tangential acceleration is constant, it may be found from the data for any two times.
\[
a_t = \frac{v_3 - v_1}{t_3 - t_1} = \frac{19 - 10}{18 - 0} = 0.5 \text{ m/s}^2
\]

Then, the linear acceleration,
\[
a = \sqrt{a_r^2 + a_t^2} = \sqrt{(0.7)^2 + (0.5)^2} = 0.8602 \text{ m/s}^2
\]

(ii) If \( \theta \) is the angle between the resultant linear acceleration and the radial acceleration,
\[
\tan \theta = \frac{a_t}{a_r} = \frac{0.5}{0.7} = 0.7142
\]
\[
\Rightarrow \quad \theta = \tan^{-1} 0.7142 = 35^\circ 32'
\]

(iii) \( a_t = \omega r \)

The angular acceleration,
\[
\alpha = \frac{a_t}{r} = \frac{0.5}{280} = 1.785 \times 10^{-3} \text{ rad/s}^2 = 1.785 \text{ mrad/s}^2
\]

(10) The frequency of revolution of a particle performing circular motion changes from 60 rpm to 180 rpm in 20 seconds. Calculate the angular acceleration of the particle. (2 marks)

Solution:

Data: \( f_1 = 60 \text{ rpm} = \frac{60}{60} \text{ rev/s} = 1 \text{ rev/s}, \)
\( f_2 = 180 \text{ rpm} = \frac{180}{60} \text{ rev/s} = 3 \text{ rev/s}, t = 20 \text{ s} \)

The angular acceleration in SI units,
\[
\alpha = \frac{\omega_2 - \omega_1}{t} = \frac{2\pi f_2 - 2\pi f_1}{t} = \frac{2\pi(3) - 2\pi(1)}{20}
\]
\[
= \frac{4\pi}{20} = 0.628 \text{ rad/s}^2 = 0.628 \text{ mrad/s}^2
\]

OR

Using non SI units, the angular frequencies are \( \omega_1 = 60 \text{ rpm} = 1 \text{ rps} \) and \( \omega_2 = 180 \text{ rpm} = 3 \text{ rps} \).
\[
\therefore \quad \alpha = \frac{\omega_2 - \omega_1}{t} = \frac{3 - 1}{20} = \frac{1}{10} = 0.1 \text{ rev/s}^2.
\]

(11) The frequency of rotation of a spinning top is 10 Hz. If it is brought to rest in 6.28 s, find the angular acceleration of a particle on its surface. (2 marks)

Solution:

Data: \( f_1 = 10 \text{ Hz}, f_2 = 0 \text{ Hz}, t = 6.28 \text{ s} \)

The angular acceleration,
\[
\alpha = \frac{\omega_2 - \omega_1}{t} = \frac{2\pi f_2 - 2\pi f_1}{t}
\]
\[
= \frac{2\pi(0) - 2\pi(10)}{6.28} = -\frac{2 \times 3.14 \times 10}{6.28} = -10 \text{ rad/s}^2
\]

(12) A wheel of diameter 40 cm starts from rest and attains a speed of 240 rpm in 4 minutes. Calculate its angular displacement in this time interval. (2 marks)

Solution:

Data: \( \omega_0 = 0, f = 240 \text{ rpm}, \frac{240}{60} = 4 \text{ rev/s}, \)
\( t = 4 \text{ min} = 240 \text{ s} \)

The angular displacement,
\[
\theta = \omega_0 t + \frac{1}{2} \alpha t^2
\]
\[
= 0 + \frac{1}{2} \left( \frac{\pi}{30} \right)(240)^2 = 960\pi \text{ rad}
\]

OR

Using non SI units, the final angular frequency \( \omega = 240 \text{ rpm} \).
\[
\therefore \quad \omega = \frac{\omega_0 t}{4} = 60 \text{ rev/min}^2
\]
\[
\Rightarrow \quad \theta = \frac{1}{2} \alpha t^2 = \frac{1}{2} \left( \frac{\pi}{30} \right)(60)(4)^2 = 480 \text{ rev}
\]

(13) A flywheel slows down uniformly from 1200 rpm to 600 rpm in 5 s. Find the number of revolutions made by the wheel in 5 s. (2 marks)

Solution:

Data: \( \omega_0 = 1200 \text{ rpm}, \omega = 600 \text{ rpm}, t = 5 \text{ s} \)

Since the flywheel slows down uniformly, its

1. ROTATIONAL DYNAMICS
angular acceleration is constant. Then, its average angular speed,
\[
\omega_{av} = \frac{\omega_i + \omega_f}{2} = \frac{1200 + 600}{2} = 900 \text{ rpm} = \frac{900 \text{ rev}}{60 \text{ s}} = 15 \text{ rps}
\]

Its angular displacement in time \( t \),
\[
\theta = \omega_{av} \cdot t = 15 \times 5 = 75 \text{ revolutions}
\]

(14) An ant is stuck to the rim of a bicycle wheel of diameter 1 m. While the bicycle is on a central stand, the wheel is set into rotation and it attains the frequency of 2 rev/s in 10 seconds, with uniform angular acceleration. Calculate (i) the number of revolutions completed by the ant in these 10 seconds (2 marks) (ii) the time taken by it for first complete revolution and the last complete revolution. (3 marks)

Solution:

Data : \( r = 0.5 \text{ m, } \omega_o = 0, \omega = 2 \text{ rps, } t = 10 \text{ s} \)

(i) Angular acceleration (\( \alpha \)) being constant, the average angular speed,
\[
\omega_{av} = \frac{\omega + \omega_i}{2} = \frac{0 + 2}{2} = 1 \text{ rps}
\]

\( \therefore \) The angular displacement of the wheel in time \( t \),
\[
\theta = \omega_{av} \cdot t = 1 \times 10 = 10 \text{ revolutions}
\]

(ii) \( \alpha = \frac{\omega - \omega_o}{t} = \frac{2 - 0}{10} = \frac{1}{5} \text{ rev/s}^2 \)

\[
\theta = \omega_o t + \frac{1}{2} \alpha t^2 = \frac{1}{2} \alpha t^2 \quad (\because \omega_o = 0)
\]

\( \therefore \) For \( t_1 = 1 \text{ rev,} \)
\[
1 = \frac{1}{2} \left( \frac{1}{5} \right) t_1^2
\]

\( \therefore \) \( t_1^2 = 10 \)

\( \therefore \) \( t_1 = \sqrt{10} \text{ s} = 3.162 \text{ s} \)

For \( t_2 = 9 \text{ rev,} \quad 9 = \frac{1}{2} \left( \frac{1}{5} \right) t_2^2 \)

\( \therefore \) \( t_2^2 = 90 \)

\( \therefore \) \( t_2 = \sqrt{90} = 3\sqrt{10} = 3(3.162) = 9.486 \text{ s} \)

The time for the last, i.e., the 10th, revolution is \( t - t_2 = 10 - 9.486 = 0.514 \text{ s} \)

Unit

1.2.2 Dynamics of circular motion : Centripetal force and centrifugal force

Q. 31. Define and explain centripetal force. (2 marks)
Ans. Definition : In the uniform circular motion of a particle, the centripetal force is the force on the particle which at every instant points radially towards the centre of the circle and produces the centripetal acceleration to move the particle in its circular path.

Explanation : A uniform circular motion is an accelerated motion, with a radially inward (i.e., centripetal) acceleration \( -\frac{v^2}{r} \) or \( -\omega^2 \mathbf{r} \), where \( \mathbf{r} \)

is the radius vector and \( \hat{r} \) is a unit vector in the direction of \( \mathbf{r} \). Hence, a net real force must act on the particle to produce this acceleration. This force, which at every instant must point radially towards the centre of the circle, is called the centripetal force.

If \( m \) is the mass of the particle, the centripetal force is \( \frac{mv^2}{r} \) or \( -m\omega^2 \mathbf{r} \).

[Notes : (1) As viewed from an inertial frame of reference, the centripetal force is necessary and sufficient for the particle to perform UCM. At any instant, if the centripetal force suddenly vanishes, the particle would fly off in the direction of its linear velocity at that instant.

(2) In case the angular or linear speed changes with time, as in nonuniform circular motion, the force is not purely centripetal but has a tangential component which accounts for the tangential acceleration.]

Q. 32. Give any two examples of centripetal force. (1 mark)
Ans. Examples of centripetal force:

(1) For an Earth-satellite in a circular orbit, the centripetal force is the gravitational force exerted by the Earth on the satellite.

(2) In the Bohr atom, the centripetal force on an electron in circular orbit around the nucleus is the attractive Coulomb force of the nucleus.

(3) When an object tied at the end of a string is revolved in a horizontal circle, the centripetal force is the tension in the string.

(4) When a car takes a turn in a circular arc on a horizontal road with constant speed, the force of static friction between the car tyres and road surfaces is the centripetal force.
Ravi

**Remember this**

*Textbook page 3*

1. **As in the case of every motion with acceleration, a net real force must act to account for the accelerated motion along a circular path.** In the case of a uniform circular motion, at every instant, this force must point towards the centre of the circular path and is called the radial or centripetal force, \( F_c \). The word ‘centripetal’ comes from the Latin for ‘centre-seeking’.

Centripetal force does not act in addition to other forces on an object. Rather, a mechanical force (like tension) or gravitational force or Coulomb force provides the centripetal force. In some cases, more than one real force act on an object and a component of their resultant is in the radial direction; centripetal is just a term describing the direction of this component.

2. **At every instant, the centripetal force \( F_c \) is perpendicular to the linear (or tangential) velocity \( \vec{v} \) (see the figure below), when the dot product \( \vec{F}_c \cdot \vec{v} \) (which is the power, or the time rate of doing work) is zero.** Hence, the centripetal force produces a centripetal acceleration (i.e., changes the direction of the velocity), but instantaneous work done by the centripetal force is always zero. Therefore, the centripetal force cannot change the linear speed and the kinetic energy of the particle.

---

**Q. 33. Define and explain centrifugal force. (2 marks)**

**Ans.** Definition: In the reference frame of a particle performing circular motion, **centrifugal force** is defined as a fictitious, radially outward force on the particle and is equal in magnitude to the particle’s mass times the centripetal acceleration of the reference frame, as measured from an inertial frame of reference.

**Explanation:** A uniform circular motion is an accelerated motion, with a centripetal acceleration of magnitude \( \omega^2/r \) or \( \omega^2r \). A frame of reference attached to the particle also has this acceleration and, therefore, is an accelerated or noninertial reference frame. The changing direction of the linear velocity appears in this reference frame as a tendency to move radially outward. This is explained by assuming a fictitious centrifugal, i.e., radially outward, force acting on the particle. Since the particle is stationary in its reference frame, the magnitude of the centrifugal force is \( mv^2/r \) or \( mv^2r \), the same as that of the centripetal force on the particle.

*Note: The word ‘centrifugal’ comes from the Latin for ‘fleeing from the centre’. The word has the same root fuge from the Latin ‘to flee’ as does refugee.*

**Q. 34. Give any two examples of centrifugal force. (1 mark)**

**Ans.** Examples of centrifugal force:

1. A person in a merry-go-round experiences a radially outward force.
2. Passengers of a car taking a turn on a level road experience a force radially away from the centre of the circular road.
3. A coin on a rotating turntable flies off for some high enough angular speed of the turntable.
4. As the Earth rotates about its axis, the centrifugal force on its particles is directed away from the axis. The force increases as one goes from the poles towards the equator. This leads to the bulging of the Earth at the equator.

**Q. 35. Only in the reference frame of a particle performing circular motion can we say that the centrifugal force on the particle balances the centripetal force. Explain. (2 marks)**

**Ans.** Refer to the answer to Q. 33.

**Q. 36. Explain why centrifugal force is called a pseudo force. (1 mark)**

**Ans.** A force which arises from gravitational, electromagnetic or nuclear interaction between matter is
called a real force. The centrifugal force does not arise due to any of these interactions. Therefore, it is not a real force.

The centrifugal force in the noninertial frame of reference of a particle in circular motion is the effect of the acceleration of the frame of reference. Therefore, it is called a pseudo or fictitious force.

Q. 37. Distinguish between centripetal force and centrifugal force. State any two points of distinction. (2 marks)

Ans.

<table>
<thead>
<tr>
<th>Centripetal force</th>
<th>Centrifugal force</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Centripetal force is the force required to provide centripetal acceleration to a particle to move it in a circular path.</td>
<td>1. The centrifugal tendency of the particle, in its accelerated, i.e., noninertial, frame of reference, is explained by assuming a centrifugal force acting on it.</td>
</tr>
<tr>
<td>2. At every instant, it is directed radially towards the centre of the circular path.</td>
<td>2. At every instant, it is directed radially away from the centre of the circular path.</td>
</tr>
<tr>
<td>3. It is a real force arising from gravitational or electromagnetic interaction between matter.</td>
<td>3. It is a pseudo force since it is the effect of the acceleration of the reference frame of the revolving particle.</td>
</tr>
</tbody>
</table>

Solution:

(1) An object of mass 0.5 kg is tied to a string and revolved in a horizontal circle of radius 1 m. If the breaking tension of the string is 50 N, what is the maximum speed the object can have? (2 marks)

Solution:

Data: \( m = 0.5 \text{ kg}, r = 1 \text{ m}, F = 50 \text{ N} \)

The maximum centripetal force that can be applied is equal to the breaking tension.

\[
\therefore \frac{mv^2}{r} = F
\]

\[
\therefore v = \sqrt{\frac{Fr}{m}} = \sqrt{\frac{50 \times 1}{0.5}} = 10 \text{ m/s}
\]

This is the maximum speed the object can have.

(2) A certain string 500 cm long breaks under a tension of 45 kg wt. An object of mass 100 g is attached to this string and whirled in a horizontal circle. Find the maximum number of revolutions that the object can make per second without breaking the string. \([g = 9.8 \text{ m/s}^2]\) (2 marks)

Solution:

Data: \( m = 100 \text{ g} = 0.1 \text{ kg}, r = 500 \text{ cm} = 5 \text{ m}, g = 9.8 \text{ m/s}^2, F = 45 \text{ kg wt} = 45 \times 9.8 \text{ N} \)

The breaking tension is equal to the maximum centripetal force that can be applied.

\[
\therefore F = m\omega^2 r
\]

But \( \omega = 2\pi f \), where \( f \) is the corresponding frequency of revolution.

\[
\therefore F = m(2\pi f)^2 r = \frac{4\pi^2 mf^2 r}{4 \times (3.142)^2 \times 0.1 \times 5}
\]

\[
\therefore f = \sqrt{\frac{F}{4\pi^2 mr}} = \sqrt{\frac{45 \times 9.8}{4 \times (3.142)^2 \times 0.1 \times 5}}
\]

The maximum number of revolutions per second, \( f = 4.726 \text{ Hz} \)

(3) A disc of radius 15 cm rotates with a speed of \( 33\frac{1}{3} \text{ rpm} \). Two coins are placed on it at 4 cm and 14 cm from its centre. If the coefficient of friction between the coins and the disc is 0.15, which of the two coins will revolve with the disc? (3 marks)

Solution:

Data: \( r = 15 \text{ cm} = 0.15 \text{ m}, f = 33\frac{1}{3} \text{ rpm} = \frac{100}{3 \times 60} \text{ rev/s} = \frac{5}{9} \text{ Hz}, \mu_s = 0.15, r_1 = 4 \text{ cm} = 0.04 \text{ m}, r_2 = 14 \text{ cm} = 0.14 \text{ m} \)

Angular speed, \( \omega = 2\pi f = 2 \times 3.142 \times \frac{5}{9} = \frac{31.42}{9} = 3.491 \text{ rad/s} \)

To revolve with the disc without slipping, the necessary centripetal force must be less than or equal to the limiting force of static friction.

Limiting force of static friction, \( f_s = \mu_s N = \mu_s (mg) \) where \( m \) is the mass of the coin and \( N = mg \) is the normal force on the coin.

\[
\therefore \mu_s g \leq \mu_s (mg)
\]

\[
\mu_s g = 0.15 \times 9.8 = 1.47 \text{ m/s}^2
\]

For the first coin, \( r_1 = 0.04 \text{ m} \).

\[
\therefore \omega^2 r_1 = (3.491)^2 \times 0.04 = 12.19 \times 0.04 = 0.4876 \text{ m/s}^2
\]
Since, $\omega^2 r_1 < \mu_s g$, this coin will revolve with the disc.

For the second coin, $r_2 = 0.14$ m.

$\therefore \omega^2 r_2 = (3.491)^2 \times 0.14 = 12.19 \times 0.14 = 1.707 \text{ m/s}^2$

Since, $\omega^2 r_2 > \mu_s g$, this coin will not revolve with the disc.

Thus, only the coin placed at 4 cm from the centre will revolve with the disc.

*(4)* The coefficient of static friction between a coin and a gramophone disc is 0.5. The radius of the disc is 8 cm. Initially, the centre of the coin is $\pi$ cm away from the centre of the disc. At what minimum frequency will it start slipping from there? By what factor will the answer change if the coin is almost at the rim? [Take $g = \pi^2 \text{ m/s}^2$] \((3 \text{ marks})\)

Solution:

Data: $\mu_s = 0.5$, $r_1 = \pi$ cm = $\pi \times 10^{-2}$ m, $r_2 = 8$ cm = $8 \times 10^{-2}$ m, $g = \pi^2 \text{ m/s}^2$

To revolve with the disc without slipping, the necessary centripetal force must be less than or equal to the limiting force of static friction.

$\therefore \omega r \leq \mu_s g$

$\therefore 4\pi^2 f_{\min}^2 = \mu_s g$ \hspace{1cm} \(\therefore \omega = 2\pi f\) \hspace{1cm} ... (1)

For $r = r_1$,

$$f_{\min,1}^2 = \frac{\mu_s g}{4\pi^2 r_1} = \frac{0.5(\pi^2)}{4\pi^2(\pi \times 10^{-2})} = \frac{100}{8\pi} \times 2\pi = \frac{25}{2\pi}$$

$\therefore f_{\min,1} = \frac{\sqrt{25}}{\sqrt{2\pi}} \text{ rps}$

The coin will start slipping when the frequency is

$$\frac{5}{\sqrt{2\pi}} \text{ rps}$$

From Eq. (1), $f_{\min,2} \propto \frac{1}{r}$

since $\mu_s$ and $g$ are constant.

$$\therefore \frac{f_{\min,2}}{f_{\min,1}} = \frac{\sqrt{r_1}}{\sqrt{r_2}} = \frac{\sqrt{\pi}}{\sqrt{8}}$$

$$\therefore f_{\min,2} = \frac{\pi}{8} f_{\min,1}$$

The minimum frequency in the second case will be $\frac{\pi}{8}$ times that in the first case.

[Note: The answers given in the textbook are for $r_1 = 2$ cm.]

---

Activity

(Textbook page 3)

Attach a body of suitable mass to a spring balance so that it stretches by about half its capacity. Now whirl the spring balance so that the body performs a horizontal circular motion. You will notice that the balance now reads more for the same body. Can you explain this?

Due to outward centrifugal force.

---

Unit

1.3 Applications of uniform circular motion

1.3.1 Vehicle along a horizontal circular track

1.3.2 Well of death

1.3.3 Vehicle along a banked circular track

Q. 39. Derive an expression for the maximum safe speed for a vehicle on a horizontal circular road without skidding off. State its significance. \((3 \text{ marks})\)

Ans. Consider a car of mass $m$ taking a turn of radius $r$ along a level road. If $\mu_s$ is the coefficient of static friction between the car tyres and the road surface, the limiting force of friction is $f_s = \mu_s N = \mu_s mg$

where $N = mg$ is the normal reaction. The forces on the car, as seen from an inertial frame of reference are shown in Fig. 1.7.

![Fig. 1.7: A car taking a circular turn on a level road](image)

Then, the maximum safe speed $v_{\max}$ with which the car can take the turn without skidding off is set by

maximum centripetal force = limiting force of static friction

$$\therefore \frac{mv_{\max}^2}{r} = \mu_s mg \quad \text{or} \quad v_{\max}^2 = r\mu_s g$$

$$\therefore v_{\max} = \sqrt{r\mu_s g}$$

This is the required expression.

Significance: The above expression shows that the maximum safe speed depends critically upon
friction which changes with circumstances, e.g., the nature of the surfaces and presence of oil or water on the road. If the friction is not sufficient to provide the necessary centripetal force, the vehicle is likely to skid off the road.

[Note: At a circular bend on a level railway track, the centrifugal tendency of the railway carriages causes the flange of the outer wheels to brush against the outer rail and exert an outward thrust on the rail. Then, the reaction of the outer rail on the wheel flange provides the necessary centripetal force.]

Do you know?
(Textbook page 4)
1. When a car takes a turn along a level road, apart from the risk of skidding off outward, it also has a tendency to roll outward due to an outward torque about the centre of gravity due to the friction force. See Fig. 1.8 in Q. 40.
2. If a bicyclist taking a turn along an unbanked road does not lean inward, an unbalanced outward torque about the centre of gravity due to the friction force will topple the bicyclist outward. The bicyclist must lean inward to counteract this torque (and not to generate a centripetal force).

Q. 40. Derive an expression for the maximum safe speed for a vehicle on a circular horizontal road without toppling/overturning/rollover. (3 marks)

Ans. Consider a car of mass \( m \) taking a turn of radius \( r \) along a level road. As seen from an inertial frame of reference, the forces acting on the car are:

1. the lateral limiting force of static friction \( f_s \) on the wheels—acting along the axis of the wheels and towards the centre of the circular path—which provides the necessary centripetal force [Fig. 1.8 (a)].
2. the weight \( mg \) acting vertically downwards at the centre of gravity (C.G.)
3. the normal reaction \( N \) of the road on the wheels, acting vertically upwards effectively at the C.G.

Since maximum centripetal force = limiting force of static friction,
\[
m a_c = \frac{m v^2}{r} = f_s
\]

In a simplified rigid-body vehicle model, we consider only two parameters—the height \( h \) of the C.G. above the ground and the average distance \( b \) between the left and right wheels called the track width.

![Fig. 1.8: Rolling tendency of a vehicle negotiating a bend on a level road](image)

The friction force \( f_s \) on the wheels produces a torque \( \tau_s \) that tends to overturn/rollover the car about the outer wheel [Fig. 1.8 (b)]. Rotation about the front-to-back axis is called roll.

\[
\tau_s = f_s \cdot h = \left( \frac{m v^2}{r} \right) h
\]

When the inner wheel just gets lifted above the ground, the normal reaction \( N \) of the road acts on the outer wheels but the weight continues to act at the C.G. Then, the couple formed by the normal reaction and the weight produces a opposite torque \( \tau_r \) which tends to restore the car back on all four wheels [Fig. 1.8 (b)]

\[
\tau_r = mg \cdot \frac{b}{2}
\]

The car does not topple as long as the restoring torque \( \tau_r \) counterbalances the toppling torque \( \tau_s \). Thus, to avoid the risk of rollover, the maximum speed that the car can have is given by

\[
\left( \frac{m v^2}{r} \right) h = mg \cdot \frac{b}{2} \Rightarrow v_{max} = \frac{rbg}{2h}
\]

Thus, vehicle tends to roll when the radial acceleration reaches a point where inner wheels of the four-wheeler are lifted off of the ground and the vehicle is rotated outward. A rollover occurs when the gravitational force \( mg \) passes through the pivot.
point of the outer wheels, i.e., the C.G. is above the line of contact of the outer wheels. Equation (3) shows that this maximum speed is high for a car with larger track width and lower centre of gravity.

**Q. 41.** (i) While driving along an unbanked circular road, a two-wheeler has to lean with the vertical. Why? (2 marks)
(ii) By what angle does the rider have to lean? Derive the relevant expression. (2 marks)
(iii) Why such a leaning is not necessary for a four-wheeler? (1 mark)

**Ans.**
(i) When a bicyclist takes a turn along an unbanked road, the force of friction $f_s$ provides the centripetal force; the normal reaction of the road $N$ is vertically up. If the bicyclist does not lean inward, there will be an unbalanced outward torque about the centre of gravity, $f_s \cdot h$, due to the friction force that will topple the bicyclist outward. The bicyclist must lean inward to counteract this torque (and not to generate a centripetal force) such that the opposite inward torque of the couple formed by $N$ and the weight $mg$, $mg \cdot a = f_s \cdot h_1$ (Fig. 1.9).

$$\tan \theta = \frac{f_s}{N} = \frac{mv^2/r}{mg} = \frac{v^2}{gr}$$

Hence, the cyclist must lean by an angle
$$\theta = \tan^{-1} \left( \frac{v^2}{gr} \right)$$

(ii) Since the force of friction provides the centripetal force,
$$f_s = \frac{mv^2}{r}$$

If the cyclist leans from the vertical by an angle $\theta$, the angle between $N$ and $F$ in Fig. 1.9 (b).

(iii) When a car takes a turn along a level road, apart from the risk of skidding off outward, it also has a tendency to roll outward due to an outward torque about the centre of gravity due to the friction force. But a car is an extended object with four wheels. So, when the inner wheels just get lifted above the ground, it can be counterbalanced by a restoring torque of the couple formed by the normal reaction (on the outer wheels) and the weight. [See Q. 40.]

**Use your brain power**

(Textbook page 4)

(i) Obtain the condition for not toppling (rollover) for a four-wheeler. On what factors does it depend and how?

Refer to the answer to Q. 40.

There will be rollover (before skidding) if $\tau_t \leq \tau_r$, that is if

$$f_s \cdot h \geq mg \cdot \frac{b}{2}$$

i.e., if $\frac{f_s}{mg} \geq \frac{b}{2h}$ or $\frac{a_r}{g} \geq \frac{b}{2h}$

or $\mu_s \geq \frac{b}{2h}$

\( \therefore \) $f_s = \mu_s N = \mu_s mg$

The vehicle parameter ratio, $\frac{b}{2h}$, is called the static stability factor (SSF). Thus, the risk of a rollover is low if SSF $\leq \mu_s$. A vehicle will most likely skid out rather than roll if $\mu_s$ is too low, as on a wet or icy road.

(ii) Think about the normal reactions. Where are those and how much are those?

In a simplified vehicle model, we assume the normal reactions to act equally on all the four wheels, i.e., $mg/4$ on each wheel. However, the C.G. is not at the wheel base, $L$. 

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1. ROTATIONAL DYNAMICS
geometric centre of a vehicle and the wheelbase (i.e., the distance \( L \) between its front and rear wheels) affects the weight distribution of the vehicle. When a vehicle is not accelerating, the normal reactions on each pair of front and rear wheels are, respectively,

\[
N_i = \frac{d_i}{L} mg \quad \text{and} \quad N_r = \frac{d_r}{L} mg
\]

where \( d_r \) and \( d_f \) are the distances of the rear and front axles from the C.G. [When a vehicle accelerates, additional torque acts on the axles and the normal reactions on the wheels change. So, as is common experience, a car pitches back (i.e., rear sinks and front rises) when it accelerates, and a car pitches ahead (i.e., front noses down). Rotation about the lateral axis is called pitch.]

(iii) What is the recommendations on loading a vehicle for not toppling easily?

Overloading (or improper load distribution) or any load placed on the roof raises a vehicle’s centre of gravity, and increases the vehicle’s likelihood of rolling over. A roof rack should be fitted by considering weight limits.

Road accidents involving rollovers show that vehicles with higher \( h \) (such as SUVs, pickup vans and trucks) topple more easily than cars. Untripped rollovers normally occur when a top-heavy vehicle attempts to perform a panic maneuver that it physically cannot handle.

(iv) If a vehicle topples while turning, which wheels leave the contact with the road? Why?

Inner wheels. Refer to the answer to Q. 40.

(v) How does [tendency to] toppling affect the tyres?

While turning, shear stress acts on the tyre-road contact area. Due to this, the treads and side wall of a tyre deform. Apart from less control, this contributes to increased and uneven wear of the shoulder of the tyres.

Each wheel is placed under a small inward angle (called camber) in the vertical plane. Under severe lateral acceleration, when the car rolls, the camber angle ensures the complete contact area is in contact with the road and the wheels are now in vertical position. This improves the cornering behavior of the car. Improperly inflated and worn tyres can be especially dangerous because they inhibit the ability to maintain vehicle control. Worn tires may cause the vehicle to slide sideways on wet or slippery pavement, sliding the vehicle off the road and increasing its risk of rolling over.

(vi) What is the recommendation for this?

Because of uneven wear of the tyre shoulders, tyres should be rotated every 10000 km-12000 km. To avoid skidding, rollover and tyre-wear, the force of friction should not be relied upon to provide the necessary centripetal force during cornering. Instead, the road surface at a bend should be banked, i.e., tilted inward. Refer Qs. 43 and 44.

(vii) Determine the angle to be made with the vertical by a two-wheeler while turning on a horizontal track?

Refer to the answer to Q. 41.

(viii) We have mentioned about ‘static friction’ between road and tyres. Why is it static friction? What about kinetic friction between road and tyres?

When a car takes a turn on a level road, the point of contact of the wheel with the surface is instantaneously stationary if there is no slipping. Hence, the lateral force on the car is the limiting force of static friction between the tyres and road. Lateral forces allow the car to turn. As long as the wheels are rolling, there is lateral force of static friction and longitudinal force of rolling friction. Longitudinal forces, which act in the direction of motion of the car body (or in the exact opposite direction), control the acceleration or deceleration of the car and therefore the speed of the car. These are the wheel force, rolling friction, braking force and air drag. If the car skids, the friction force is kinetic friction; more importantly, the direction of the friction force then changes abruptly from lateral to that opposite the velocity of skidding and not towards the centre of the curve, so that the car cannot continue in its curved path.
(ix) What do you do if your vehicle is trapped on a slippery or sandy road? What is the physics involved?

Driving on a country road should be attempted only with a four-wheel drive. However, if you do get stuck in deep sand or mud, avoid unnecessary panic and temptation to drive your way out of the mud or sand because excessive spinning of your tyres will most likely just dig you into a deeper hole. Momentum is the key to getting unstuck from sand or mud. One method is the rocking method—rocking your car backwards and forwards to gain momentum. Your best option is usually to gain traction and momentum by wedging a car mat (or sticks, leaves, gravel or rocks) in front and under your drive wheels. Once you start moving, keep the momentum going until you are on more solid terrain.

Q. 42. A carnival event known as a "well of death" consists of a large vertical cylinder inside which usually a stunt motorcyclist rides in horizontal circles. Show that the minimum speed necessary to keep the rider from falling is given by

\[ v = \sqrt{rg/\mu_s} \] in usual notations. (3 marks)

Ans. The forces exerted on the rider are

(i) the normal force \( \vec{N} \) exerted by the wall, directed radially inward, is the centripetal force,

(ii) the upward frictional force \( \vec{f}_s \) exerted by the wall, since the motorcycle has a tendency to slide down,

(iii) the downward gravitational force \( mg \).

\[ \therefore N = \frac{mv^2}{r} \quad \text{and} \quad f_s = \mu_s N = \mu_s \left( \frac{mv^2}{r} \right) \]

where \( m \) is the mass of the rider and \( v \) is the speed of the motorcyclist. For the rider not to fall, \( f_s \) must balance \( g \).

\[ \therefore \mu_s \left( \frac{mv^2}{r} \right) = mg \quad \therefore v^2 = \frac{rg}{\mu_s} \quad \therefore v = \sqrt{\frac{rg}{\mu_s}} \]

which is the required expression.

Remember this (Textbook page 5)

In a well-of-death, \( f_s = mg \) and \( N = \frac{mv^2}{r} \) are valid for any value of \( v \) but \( f_s = \mu_s N \) only for \( v_{\text{min}} \). Therefore,

\[ N_{\text{min}} = \frac{mv_{\text{min}}^2}{r} = mg\mu_s. \]

Also, in the discussion, we had assumed the vehicle to be a point mass. In reality, a two-wheeler cannot be exactly perpendicular to the vertical wall or else the torque due to the couple \( mg \) and \( f_s \) will roll the vehicle down. A two-wheeler must lean as shown so that the torque of \( \vec{N} \) about the C.G. balances the propensity to roll. When a four-wheeler is to be driven in a well-of-death, its walls are never vertical but tilted outward for the same reason.

Q. 43. Explain why a road at a bend should be banked. What is angle of banking? OR What is banking of a road? Why is it necessary? OR ★ Why are curved roads banked? (2 marks)

Ans. A car while taking a turn performs circular motion. If the road is level (or horizontal road), the necessary centripetal force is the force of static friction between the car tyres and the road surface.

The friction depends upon the nature of the surfaces in contact and the presence of oil and water
on the road. If the friction is inadequate, a speeding car may skid off the road. Since the friction changes with circumstances, it cannot be relied upon to provide the necessary centripetal force. Moreover, friction results in fast wear and tear of the tyres.

To avoid the risk of skidding as well as to reduce the wear and tear of the car tyres, the road surface at a bend is tilted inward, i.e., the outer side of the road is raised above its inner side. This is called banking of road. On a banked road, the resultant of the normal reaction and the gravitational force can act as the necessary centripetal force. Thus, every car can be safely driven on such a banked curve at certain optimum speed, without depending on friction. Hence, a road should be properly banked at a bend.

The angle of banking is the angle of inclination of the surface of a banked road at a bend with the horizontal.

**Q. 44. Do we need a banked road for a two-wheeler?**
**Explain.** (1 mark)
**Ans.** When a two-wheeler takes a turn along an unbanked road, the force of friction provides the centripetal force. The two-wheeler leans inward to counteract a torque that tends to topple it outward. Firstly, friction cannot be relied upon to provide the necessary centripetal force on all road conditions. Secondly, the friction results in wear and tear of the tyres. On a banked road at a turn, any vehicle can negotiate the turn without depending on friction and without straining the tyres.

**Q. 45. A road at a bend should be banked for an optimum or most safe speed \( v_o \). Derive an expression for the required angle of banking.**
**OR**
Obtain an expression for the optimum or most safe speed with which a vehicle can be driven along a curved banked road. Hence show that the angle of banking is independent of the mass of a vehicle. (4 marks)
**Ans.** Consider a car taking a left turn along a road of radius \( r \) banked at an angle \( \theta \) for a designed optimum or most safe speed \( v_o \). Let \( m \) be the mass of the car. In general, the forces acting on the car are
(a) its weight \( mg \), acting vertically down
(b) the normal reaction of the road \( N \), perpendicular to the road surface
(c) the frictional force \( f_s \) along the inclined surface of the road.
At the optimum speed, frictional force is not relied upon to contribute to the necessary lateral centripetal force. Thus, ignoring \( f_s \), resolve \( N \) into two perpendicular components: \( N \cos \theta \) vertically up and \( N \sin \theta \) horizontally towards the centre of the circular path. Since there is no acceleration in the vertical direction, \( N \cos \theta \) balances \( mg \) and \( N \sin \theta \) provides the necessary centripetal force.

\[ N \sin \theta = \frac{mv_o^2}{r} \quad \ldots \ (1) \]
\[ N \cos \theta = mg \quad \ldots \ (2) \]
Dividing Eq. (1) by Eq. (2),
\[ \frac{N \sin \theta}{N \cos \theta} = \frac{mv_o^2/r}{mg} \]
\[ \therefore \tan \theta = \frac{v_o^2}{rg} \quad \text{or} \quad \theta = \tan^{-1} \left( \frac{v_o^2}{rg} \right) \quad \ldots \ (3) \]
Equation (3) gives the expression for the required angle of banking. From Eq. (3), we can see that \( \theta \) depends upon \( v_o \), \( r \) and \( g \). The angle of banking is independent of the mass of a vehicle negotiating the curve. Also, for a given \( r \) and \( \theta \), the recommended optimum speed is
\[ v_o = \sqrt{rg \tan \theta} \quad \ldots \ (4) \]

**Q. 46. State any two factors on which the most safe speed of a car in motion along a banked road depends.** (1 mark)
**Ans.** The angle of banking of the road and the radius of the curved path.
Q. 47. A curved horizontal road must be banked at an angle \( \theta' \) for an optimum speed \( v \). What will happen to a vehicle moving with a speed \( v \) along this road if the road is banked at an angle \( \theta \) such that (i) \( \theta < \theta' \) (ii) \( \theta > \theta' \)? (1 mark each)

Ans.
(i) For \( \theta < \theta' \), the horizontal component of the normal reaction would be less than the optimum value and will not be able to provide the necessary centripetal force. Then, the vehicle will tend to skid outward, up the inclined road surface.
(ii) For \( \theta > \theta' \), the horizontal component of the normal reaction would be more than the necessary centripetal force. Then, the vehicle will tend to skid down the banked road.

Q. 48. A banked circular road is designed for traffic moving at an optimum or most safe speed \( v_o \). Obtain an expression for (a) the minimum safe speed (b) the maximum safe speed with which a vehicle can negotiate the curve without skidding. (4 marks each)

Ans. Consider a car taking a left turn along a road of radius \( r \) banked at an angle \( \theta \) for a designed optimum speed \( v \). Let \( m \) be the mass of the car. In general, the forces acting on the car are
(a) its weight \( mg \), acting vertically down
(b) the normal reaction of the road \( N \), perpendicular to the road surface
(c) the frictional force \( f_s \) along the inclined surface of the road.

If \( \mu_s \) is the coefficient of static friction between the tyres and road, \( f_s = \mu_s N \).

(a) For minimum safe speed: If the car is driven at a speed less than the optimum speed \( v_o \), it may tend to slide down the inclined surface of the road so that \( f_s \) is up the incline, Fig. 1.12.

Resolve \( N \) and \( f_s \) into two perpendicular components: \( N \cos \theta \) and \( f_s \sin \theta \) vertically up; \( N \sin \theta \) horizontally towards the centre of the circular path while \( f_s \cos \theta \) horizontally outward. So long as the car takes the turn without sliding down, the sum \( N \cos \theta + f_s \sin \theta \) balances \( mg \), and \( N \sin \theta - f_s \cos \theta \) provides the necessary centripetal force. If \( v_{\text{min}} \) is the minimum safe speed without skidding,

\[
\frac{mv_{\text{min}}^2}{r} = N \sin \theta - f_s \cos \theta = (N \sin \theta - \mu_s \cos \theta) \quad \text{... (1)}
\]

and

\[
mg = N \cos \theta + f_s \sin \theta = (N \cos \theta + \mu_s \sin \theta) \quad \text{... (2)}
\]

\[
\therefore \quad v_{\text{min}} = \sqrt{\frac{rg(\tan \theta - \mu_s)}{1 + \mu_s \tan \theta}} \quad \text{... (3)}
\]

For \( \tan \theta \leq \mu_s \) (as on most rough roads), \( v_{\text{min}} = 0 \) (i.e., a car can be brought to a halt without sliding down).

(b) For maximum safe speed: If the car is driven fast enough, at a speed greater than the optimum speed \( v \), it may skid off up the incline so that \( f_s \) is down the incline, Fig. 1.13.

Resolve \( N \) and \( f_s \) into two perpendicular components: \( N \cos \theta \) vertically up and \( f_s \sin \theta \) vertically down; \( N \sin \theta \) and \( f_s \cos \theta \) horizontally towards the
centre of the circular path. So long as the car takes
the turn without skidding off, the horizontal
components $N\sin \theta$ and $f_s \cos \theta$ together provide
the necessary centripetal force, and $N \cos \theta$ balances
the sum $mg + f_s \sin \theta$. If $v_{\text{max}}$ is the maximum safe
speed without skidding,
\[
\frac{mv^2_{\text{max}}}{r} = N \sin \theta + f_s \cos \theta
\]
\[
= N \sin \theta + \mu_s N \cos \theta
\]
\[
= N (\sin \theta + \mu_s \cos \theta)
\] ... (4)
and $N \cos \theta = mg + f_s \sin \theta$
\[
= mg + \mu_s N \sin \theta
\] \[
\therefore \ mg = N (\cos \theta - \mu_s \sin \theta)
\] ... (5)
Dividing Eq. (4) by Eq. (5),
\[
\frac{mv^2_{\text{max}}/r}{mg} = \frac{N \sin \theta + \mu_s \cos \theta}{N \cos \theta - \mu_s \sin \theta}
\]
\[
\therefore \ v_{\text{max}} = \frac{\sin \theta + \mu_s \cos \theta}{\cos \theta - \mu_s \sin \theta} \tan \theta + \mu_s
\]
\[
= \frac{\tan \theta + \mu_s}{1 - \mu_s \tan \theta}
\] ... (6)
For $1 - \mu_s \tan \theta = 0$, i.e., $\mu_s = \cot \theta$, $v_{\text{max}} = \infty$.

Ignoring few special cases, the maximum value of
$\mu_s = 1$. Thus, for $\theta \geq 45^\circ$, $v_{\text{max}} = \infty$, i.e., on a heavily
banked road a car is unlikely to skid up the incline and the minimum limit is more important.

Use your brain power

(Textbook page 6)

As a civil engineer, you are to construct a curved road
in a ghat. In order to calculate the banking angle $\theta$,
you need to decide the speed limit. How will you
decide the values of speed and radius of curvature at
the bend?

For Indian roads, Indian Road Congress (IRC),
[IRC-73-1980, Table 2, p.4], specifies the design
speed depending on the classification of roads (such
as national and state highways, district roads and
village roads) and terrain. It is the basic design
parameter which determines further geometric
design features. For the radius of curvature at a bend,
IRC [ibid., Table 16, p.24] specifies the absolute
minimum values based on the minimum design
speed. However, on new roads, curves should be
designed to have the largest practicable radius,
generally more than the minimum values specified.

to allow for ‘sight distance’ and ‘driver comfort’. To
consider the motorist driving within the innermost
travel lane, the radius used to design horizontal
curves should be measured to the inside edge of the
innermost travel lane, particularly for wide road-
ways with sharp horizontal curvature.

A civil engineer refers to banking as superelevation
$e$; $e = \tan \theta$. IRC fixes $e_{\text{max}} = 0.07$ for a non-urban road
and the coefficient of lateral static friction, $\mu = 0.15$,
the friction between the vehicle tyres and the road
being incredibly variable. Ignoring the product $e\mu$,
from Eq. (6) in Q. 48.
\[
e + \mu = \frac{v^2}{gr}
\] (where both $v$ and $r$ are in SI units)
\[
e = \frac{V^2}{127r}
\] (where $V$ is in km/h and $r$ is in metre)
... (1)
The sequence of design usually goes like this:

1. Knowing the design speed $V$ and radius $r$, calculate
the superelevation for 75% of design speed
ignoring friction: $e = (0.75V)^2 \frac{V^2}{127r} = \frac{V^2}{225r}$

2. If $e < 0.07$, consider this calculated value of $e$ in
subsequent calculations. If $e > 0.07$, then take
$e = e_{\text{max}} = 0.07$.

3. Use Eq. (1) above to check the value of $\mu$ for
$e_{\text{max}} = 0.07$ at the full value of the design speed $V$:
\[
\mu = \frac{V^2}{127r} - 0.07.
\]
If $\mu < 0.15$, then $e = 0.07$ is safe. Otherwise, calculate
the allowable speed $V_a$ as in step 4.

4. $V_a = e + \mu = 0.07 + 0.15$
If $V_a > V$, then the design speed $V$ is adequate.
If $V_a < V$, then speed is limited to $V_a$ with
appropriate warning sign.

Use your brain power

(Textbook page 7)
If friction is zero, can a vehicle move on the road?
Why are we not considering the friction in deriving
the expression for the banking angle?
Friction is necessary for any form of locomotion.
Without friction, a vehicle cannot move.
The banking angle for a road at a bend is calculated
for optimum speed at which every vehicle can negotiate the bend without depending on friction to provide the necessary lateral centripetal force.

What about the kinetic friction between the road and the tyres?

Refer to the explanation to Q. (viii) in the Use your brain power box of textbook page 4.

Solved Problems 1.3 – 1.3.3

(Data :  

Q. 49. Solve the following :

(1) Find the maximum speed with which a car can be safely driven along a curve of radius 100 m, if the coefficient of friction between its tyres and the road is 0.2 \( [g = 9.8 \text{ m/s}^2] \). (2 marks)

\[ v = \sqrt{r\mu_s g} \]
\[ \therefore \mu_s \text{ and } g \text{ remaining constant, } v \propto \sqrt{r} \]

\[ \frac{v_2}{v_1} = \sqrt{\frac{r_2}{r_1}} = \sqrt{\frac{2r}{r}} = \sqrt{2} \]
\[ \therefore v_2 = \sqrt{2}v_1 = \sqrt{2}v \]

The maximum safe speed of the car on the second road is \( \sqrt{2}v \).

(4) On a dry day, the maximum safe speed at which a car can be driven on a curved horizontal road without skidding is 7 m/s. When the road is wet, the frictional force between the tyres and road reduces by 25%. How fast can the car safely take the turn on the wet road? (2 marks)

Solution : Let subscripts 1 and 2 denote the values of a quantity under dry and wet conditions, respectively.

Data : \( v_1 = 7 \text{ m/s}, f_2 = f_1 - 0.25f_1 = 0.75f_1 \)

On a dry horizontal curved road, the frictional force between the tyres and road is \( f_1 = \mu_1 mg \), where \( m \) is the mass of the car and \( g \) is the gravitational acceleration.

The maximum safe speed for taking a turn of radius \( r \) on a dry horizontal curved road is

\[ v_1 = \sqrt{\mu_1 rg} = \sqrt{\frac{r}{m}} \sqrt{f_1} \]

If the road is wet, the corresponding quantities are

\[ f_2 = \mu_2 mg \text{ and } v_2 = \sqrt{\frac{r}{m}} \sqrt{f_2} \]

Thus, for \( m \) and \( r \) remaining the same,

\[ \frac{v_1}{v_2} = \sqrt{\frac{f_1}{f_2}} \]
\[ \therefore v_2 = \frac{f_2}{f_1} \cdot v_1 = \frac{0.75f_1}{f_1} \cdot \sqrt{\frac{r}{m}} \cdot \sqrt{f_1} \]
\[ = 7 \times 0.75 = 7 \times 0.866 = 6.062 \text{ m/s} \]

(5) A coin kept at a distance of 5 cm from the centre of a turntable of radius 1.5 m just begins to slip when the turntable rotates at a speed of 90 rpm. Calculate the coefficient of static friction between the coin and the turntable. \( [g = \pi^2 \text{ m/s}^2] \) (2 marks)
Solution:

Data: \( r = 5 \text{ cm} = 0.05 \text{ m}, f = 90 \text{ rpm} = \frac{90}{60} \text{ rps} = 1.5 \text{ rps}, g = \pi^2 \text{ m/s}^2 \)

The centripetal force for the circular motion of the coin is provided by the friction between the coin and the turntable. The coin is just about to slip off the turntable when the limiting force of friction is equal to the centripetal force.

\[
\therefore \mu_s mg = \frac{mv^2}{r}
\]

The coefficient of static friction,

\[
\mu_s = \frac{v^2}{rg} = \frac{(2\pi f)^2 \times r}{rg} = \frac{\omega^2 r}{g} = \frac{(2\pi f)^2}{g} \quad (\therefore \omega = 2\pi f)
\]

\[
= \frac{4\pi^2 f^2 r}{g} = \frac{4\pi^2 \times (1.5)^2 \times 0.05}{g} = 0.2 \times 2.25 = 0.45
\]

(6) A thin cylindrical shell of inner radius 1.5 m rotates horizontally about a vertical axis, at an angular speed \( \omega \). A wooden block rests against the inner surface and rotates with it. If the coefficient of static friction between block and surface is 0.3, how fast must the shell be rotating if the block is not to slip and fall? (2 marks)

Solution:

Data: \( r = 1.5 \text{ m}, \mu_s = 0.3 \)

The normal force \( N \) of the shell on the block is the centripetal force which holds the block in place. \( N \) determines the friction on the block, which in turn keeps it from sliding downward. If the block is not to slip, the friction force \( f_s \) must balance the weight \( mg \) of the block.

\[
\therefore N = m\omega^2 r \quad \text{and} \quad f_s = \mu_s N = mg
\]

\[
\therefore \mu_s (m\omega^2 r) = mg
\]

(7) A motorcyclist rounds a curve of radius 25 m at 36 km/h. The combined mass of the motorcycle and the man is 150 kg. (i) What is the centripetal force exerted on the motorcyclist? (ii) What is the upward force exerted on the motorcyclist? (2 marks)

Solution:

Data: \( r = 25 \text{ m}, v = 36 \text{ km/h} = 36 \times \frac{5}{18} \text{ m/s} = 10 \text{ m/s}, m = 150 \text{ kg}, g = 10 \text{ m/s}^2 \)

(i) Centripetal force, \( F = \frac{mv^2}{r} = \frac{150 \times (10)^2}{25} = 600 \text{ N} \)

(ii) Upward force = normal reaction of the road surface = \( mg = 150 \times 10 = 1500 \text{ N} \)

(8) A motorcyclist is describing a circle of radius 25 m at a speed of 5 m/s. Find his inclination with the vertical. What is the value of the coefficient of friction between the tyres and ground? (2 marks)

Solution:

Data: \( v = 5 \text{ m/s}, r = 25 \text{ m}, g = 10 \text{ m/s}^2 \)

(i) \( \tan \theta = \frac{v^2}{rg} = \frac{(5)^2}{25 \times 10} = 0.10 \)

\[
\therefore \theta = \tan^{-1} 0.10 = 5^\circ 4' \quad \text{(incline with the vertical)}
\]

(ii) \( \frac{mv^2}{r} = \mu_s mg \)

where \( \mu_s \) is the coefficient of friction.

\[
\therefore \mu_s = \frac{v^2}{rg} = 0.10
\]

(9) A motor van weighing 4400 kg (i.e., a motor van of mass 4400 kg) rounds a level curve of radius 200 m on an unbanked road at 60 km/h. What should be the minimum value of the coefficient of friction to prevent skidding? At what angle should the road be banked for this velocity? (3 marks)

Solution:

Data: \( m = 4400 \text{ kg}, r = 200 \text{ m}, v = 60 \text{ km/h} = 60 \times \frac{5}{18} \text{ m/s} = \frac{50}{3} \text{ m/s}, g = 10 \text{ m/s}^2 \)
The minimum value of the coefficient of friction is 
\[ \mu_s = \frac{v^2}{rg} = \frac{(50/3)^2}{200 \times 10} = \frac{25}{18 \times 10} = 0.1389 \]

The angle of banking, \( \theta = \tan^{-1} 0.1389 \) 
\[ \approx 7^\circ 5' \]

[Note: In part (ii), \( v \) is to be taken as the optimum speed.]

**Solution:**

**Data:** \( R = 4 \) m, \( \mu_s = 0.4, g = 10 \) m/s\(^2\)

The forces exerted on the rider, when the floor drops away, are

(i) the normal force \( N \) exerted by the wall, directed radially inward, is the centripetal force

(ii) the upward frictional force \( f_s \) exerted by the wall

(iii) the downward gravitational force \( mg \)

\[ N = m\omega^2R \text{ and } f_s = \mu_s N = \mu_s (m\omega^2R) \] where \( m \) is the mass of the rider and \( \omega \) is the angular speed of the Rotor cylinder. For the rider not to fall, \( f_s \) must balances \( mg \).

\[ \mu_s (m\omega^2R) = mg \]

\[ \omega^2 = \frac{g}{\mu_s R} \]

\[ \omega = \sqrt{\frac{g}{\mu_s R}} \]

This is the minimum angular speed necessary.

Since \( \omega = 2\pi f \), the corresponding frequency of rotation of the cylinder is

\[ f = \frac{1}{2\pi} \sqrt{\frac{g}{\mu_s R}} \]

\[ = \frac{1}{2 \times 3.142} \sqrt{\frac{10}{0.4 \times 4}} = \frac{1}{6.284} \sqrt{\frac{100}{16}} \]

\[ = \frac{10}{6.284 \times 4} = 0.3978 \text{ Hz} \]

\[ = 0.3978 \times 60 = 23.87 \text{ rpm} \]

(12) The two rails of a broad-gauge railway track are 1.68 m apart. At a circular curve of radius 1.6 km, the outer rail is raised relative to the inner rail by 8.4 cm. Find the angle of banking of the track and the corresponding frequency of rotation of the cylinder.

\[ \cos \theta = \frac{H}{R} = \frac{8.4}{1600} \]

\[ \theta = \cos^{-1} \left( \frac{8.4}{1600} \right) \]

\[ \approx 90^\circ - 8.4^\circ \]

\[ = 81.6^\circ \]

(3 marks)
the optimum speed of a train rounding the curve.

(3 marks)

Solution:

Data: \( l = 1.68 \text{ m} = 168 \text{ cm}, \ r = 1.6 \text{ km} = 1600 \text{ m}, \ h = 8.4 \text{ cm}, \ g = 10 \text{ m/s}^2 \)

(1) If \( \theta \) is the banking angle,

\[
\sin \theta = \frac{h}{l} = \frac{8.4}{168} = 0.05
\]

\[
\therefore \theta = 2^\circ 52'\]

(2) \( \tan \theta = \frac{v^2}{rg} \)

\[
\therefore \ v^2 = rg \tan \theta
\]

\[
= (1600) (10) \tan 2^\circ 52' \\
= 1600 \times 10 \times 0.0501 = 801.6
\]

The optimum speed \( v = \sqrt{801.6} \)

\[
= 28.31 \text{ m/s} = 101.9 \text{ km/h}
\]

(14) Part of a racing track is to be designed with radius of curvature 72 m. We are not recommending the vehicles to drive faster than 216 kmph. With what angle should the road be tilted? By what height will its outer edge be with respect to the inner edge if the track is 10 m wide? (3 marks)

Solution:

Data: \( r = 72 \text{ m}, v_o = 216 \text{ km/h} = 216 \times \frac{5}{18} \)

\[
= 60 \text{ m/s}, w = 10 \text{ m}, g = 10 \text{ m/s}^2
\]

\[
\tan \theta = \frac{v_o^2}{rg} = \frac{(60)^2}{720} = \frac{3600}{720} = 5
\]

\[
\therefore \theta = \tan^{-1} 5 = 78^\circ 4'
\]

This is the required angle of banking.

\[
\sin \theta = \frac{h}{w}
\]

\[
\therefore h = w \sin \theta = (10) \sin 78^\circ 4' = 10 \times 0.9805
\]

\[
= 9.805 \text{ m}
\]

This gives the height of the outer edge of the track relative to the inner edge.

(15) A circular race course track has a radius of 500 m and is banked at 10\(^\circ\). The coefficient of static friction between the tyres of a vehicle and the road surface is 0.25. Compute (i) the maximum speed to avoid slipping (ii) the optimum speed to avoid wear and tear of the tyres. (2 marks each)

Solution:

Data: \( r = 500 \text{ m}, \ \theta = 10^\circ, \ \mu_s = 0.25, \ g = 9.8 \text{ m/s}^2, \ tan 10^\circ = 0.1763 \)

(i) On the banked track, the maximum speed of the vehicle without slipping (skidding) is

\[
v_{max} = \frac{rg (\mu_s + \tan \theta)}{\sqrt{1 - \mu_s \tan \theta}}
\]

\[
= \frac{500 \times 10 (0.25 + 0.1763)}{\sqrt{1 - (0.25 \times 0.1763)}}
\]

\[
= \frac{500 \times 10 \times 0.4263}{0.9559} = \sqrt{2230}
\]

\[
= 47.22 \text{ m/s}
\]

(ii) The optimum speed of the vehicle on the track is

\[
v_{opt} = \sqrt{rg \tan \theta}
\]

\[
= \sqrt{500 \times 10 \times 0.1763}
\]

\[
= \sqrt{881.5} = 29.69 \text{ m/s}
\]

(16) The track in problem (14) is constructed as per the requirements. The coefficient of static friction between the tyres of a vehicle on this track is 0.8, will there be any lower speed limit? By how much can the upper speed limit exceed in this case? (3 marks)

Solution:

Data: \( r = 72 \text{ m}, \ \theta = 78^\circ 4', \ \mu_s = 0.8, \ g = 10 \text{ m/s}^2 \)

\[
\tan \theta = \tan 78^\circ 4' = 5
\]
\[ v_{\min} = \sqrt{rg\left(\frac{\tan \theta - \mu_s}{1 + \mu_s \tan \theta}\right)} \]

\[ = \sqrt{(72)(10)\left(\frac{5 - 0.8}{1 + (0.8)(5)}\right)} \]

\[ = \sqrt{720 \times \frac{4.2}{5}} = \sqrt{144 \times 4.2} = 12 \times 2.049 \]

\[ = 24.588 \text{ m/s} = 88.52 \text{ km/h} \]

This will be the lower limit or minimum speed on this track.

Since the track is heavily banked, \( \theta > 45^\circ \), there is no upper limit or maximum speed on this track.

**Unit 1.3.4 Conical pendulum**

Q. 50. Define a conical pendulum. \( (1 \text{ mark}) \)
Ans. A **conical pendulum** is a small bob suspended from a string and set in UCM in a horizontal plane with the centre of its circular path below the point of suspension such that the string makes a constant angle \( \theta \) with the vertical.

**OR**

A **conical pendulum** is a simple pendulum whose bob revolves in a horizontal circle with constant speed such that the string describes the surface of an imaginary right circular cone.

Q. 51. Derive an expression for the angular speed of the bob of a conical pendulum. \( (3 \text{ marks}) \) OR
Derive an expression for the frequency of revolution of the bob of a conical pendulum. \( (4 \text{ marks}) \)
Ans. Consider a conical pendulum of string length \( L \) with its bob of mass \( m \) performing UCM along a circular path of radius \( r \) (Fig. 1.16).

At every instant of its motion, the bob is acted upon by its weight \( mg \) and the tension \( F \) in the string.
If the constant angular speed of the bob is \( \omega \), the necessary horizontal centripetal force is

\[ F_c = m\omega^2 r \]

\( F_c \) is the resultant of the tension in the string and the weight. Resolve \( F \) into components \( F \cos \theta \) vertically opposite to the weight of the bob and \( F \sin \theta \) horizontal. \( F \cos \theta \) balances the weight. \( F \sin \theta \) is the necessary centripetal force.

\[ \therefore F \sin \theta = m\omega^2 r \quad \ldots (1) \]
and \( F \cos \theta = mg \quad \ldots (2) \)

Dividing Eq. (1) by Eq. (2),
\[ \tan \theta = \frac{\omega^2 r}{g} \]

From the diagram,
\[ \tan \theta = \frac{r}{OC} = \frac{r}{h} = \frac{r}{L \cos \theta} \quad \ldots (3) \]

\[ \therefore \frac{r}{h} = \frac{\omega^2 r}{g} \]
\[ \therefore \omega^2 = \frac{g}{h} = \frac{g}{L \cos \theta} \quad \ldots (4) \]

The angular speed of the bob,
\[ \omega = \sqrt{\frac{g}{h}} = \sqrt{\frac{g}{L \cos \theta}} \quad \ldots (5) \]
is the required expression for \( \omega \).

[**Note:** From Eq. (4), \( \cos \theta = g/\omega^2 L \). Therefore, as \( \omega \) increases, \( \cos \theta \) decreases and \( \theta \) increases.]

If \( n \) is the frequency of revolution of the bob,
\[ \omega = 2\pi n = \sqrt{\frac{g}{L \cos \theta}} \]
\[ \therefore n = \frac{1}{2\pi} \sqrt{\frac{g}{L \cos \theta}} \quad \ldots (6) \]
is the required expression for the frequency.

Q. 52. What will happen to the angular speed of a conical pendulum if its length is increased from 0.5 m to 2 m, keeping other conditions the same? \( (1 \text{ mark}) \)
Ans. The angular speed of the conical pendulum will become half the original angular speed.

**Q. 53.** On what factors does the frequency of conical pendulum depend? Is it independent of some factors? \( (2 \text{ marks}) \)
Ravi

The frequency of a conical pendulum, of string length $L$ and semivertical angle $\theta$, is

$$n = \frac{1}{2\pi} \sqrt{\frac{g}{L \cos \theta}}$$

where $g$ is the acceleration due to gravity at the place.

From the above expression, we can see that

(i) $n \propto \sqrt{\frac{g}{L \cos \theta}}$

(ii) $n \propto \frac{1}{\sqrt{L}}$

(iii) $n \propto \frac{1}{\sqrt{\cos \theta}}$

(if $\theta$ increases, $\cos \theta$ decreases and $n$ increases)

(iv) The frequency is independent of the mass of the bob.

Q. 54. Define period of a conical pendulum and obtain an expression for it. (3 marks)

Ans. The period of a conical pendulum is the time taken by its bob to complete one revolution in a horizontal circle with constant speed.

For the derivation, refer to the answer to Q. 51 up to Eq. (5) and continue:

If $T$ is the period, $\omega = \frac{2\pi}{T}$

\[T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{L \cos \theta}{g}}\]

is the required expression.

[Note : $L \cos \theta = OC = h$, where $h$ is the axial height of the cone.

\[T = 2\pi \sqrt{\frac{h}{g}}\]

This shows that the period of a conical pendulum is the same as that of a simple pendulum of length $h$.]

Q. 55. Write an expression for the time period of a conical pendulum. State how the period depends on the various factors. (2 marks)

Ans. If $T$ is the time period of a conical pendulum of string length $L$ which makes a constant angle $\theta$ with the vertical,

$$T = 2\pi \sqrt{\frac{L \cos \theta}{g}}$$

where $g$ is the acceleration due to gravity at the place.

From the above expression, we can see that

(i) $T \propto \sqrt{L}$

(ii) $T \propto \sqrt{\cos \theta}$

(if $\theta$ increases, $\cos \theta$ and $T$ decrease)

(iii) $T \propto \frac{1}{\sqrt{g}}$

(iv) The period is independent of the mass of the bob.

Do you know? (Textbook page 8)

Conical pendulum

- For a given conical pendulum of string length $L$ at a place where gravitational acceleration is $g$, its time period and frequency depend upon $\theta$. If $\theta$ increases, $\cos \theta$ and $T$ decrease while $n$ increases.

- The period $T \propto \sqrt{\cos \theta}$, the tension $F \propto \tan \theta$, and the linear speed $v \propto \sqrt{\tan \theta}$.

Thus, as the angle $\theta$ increases, the $\cos \theta$ decreases and $\tan \theta$ increases. For $\theta = 90^{\circ}$, $T = 0$, $F = \infty$ and $v = \infty$.

In practice, the limiting value of $\theta$ will depend on the breaking tension of the string, and a body tied to a string cannot be revolved in a horizontal circle such that the string is horizontal.

Activity (Textbook page 8)

A stone is tied to a string and whirled such that the stone performs horizontal circular motion. It can be seen that the string is NEVER horizontal.

Activity (Textbook page 9)

Using a funnel and a marble or a ball bearing try to work out the situation in the (Textual Example 1.5). Try to realize that as the marble goes towards the rim of the funnel its linear speed increases but its angular speed decreases. When nearing the base, it is the other way.

Solved Problems 1.3.4

[Data : $g = 10 \text{ m/s}^2$ unless specified otherwise]

Q. 56. Solve the following :

(1) A stone of mass 2 kg is whirled in a horizontal circle attached at the end of a 1.5 m long string. If
the string makes an angle of 30° with the vertical, compute its period. (2 marks)

Solution:

Data: \( L = 1.5 \text{ m}, \theta = 30^\circ, g = 10 \text{ m/s}^2 \)

The period of the conical pendulum,

\[
T = 2\pi \sqrt{\frac{L \cos \theta}{g}} = 2 \times 3.142 \times \sqrt{\frac{1.5 \cos 30^\circ}{10}}
\]

\[
= 6.284 \times \sqrt{\frac{1.5 \times 0.866}{10}} = 6.284 \times \sqrt{1.299} = 2.265 \text{ s}
\]

(2) A string of length 0.5 m carries a bob of mass 0.1 kg at its end. If this is to be used as a conical pendulum of period 0.4 s, calculate the angle of inclination of the string with the vertical and the tension in the string. (3 marks)

Solution:

Data: \( L = 0.5 \text{ m}, m = 0.1 \text{ kg}, T = 0.4 \text{ s}, g = 10 \text{ m/s}^2 \)

(i) Period,

\[
T = 2\pi \sqrt{\frac{L \cos \theta}{g}}
\]

\[
\therefore \cos \theta = \frac{gT^2}{4\pi^2L}
\]

\[
= \frac{10(0.4)^2}{4 \times 3.14^2 \times 0.5}
\]

\[
= \frac{10 \times 0.16}{2 \times 3.14^2} = 0.08 \text{ or } 8^\circ
\]

The inclination of the string with the vertical, \( \theta = 36^\circ 8' \)

(ii) The tension in the string,

\[
F = \frac{mg}{\cos \theta} = \frac{0.1 \times 10}{0.8} = 1.25 \text{ N}
\]

(3) In a conical pendulum, a string of length 120 cm is fixed at a rigid support and carries a bob of mass 150 g at its free end. If the bob is revolved in a horizontal circle of radius 0.2 m around a vertical axis, calculate the tension in the string. \( g = 9.8 \text{ m/s}^2 \) (3 marks)

Solution:

Data: \( L = 120 \text{ cm} = 1.2 \text{ m}, m = 0.15 \text{ kg}, r = 0.2 \text{ m}, g = 9.8 \text{ m/s}^2 \)

\[
\sin \theta = \frac{r}{L} = \frac{0.2}{1.2} = \frac{1}{6}
\]

\[
\theta = \sin^{-1}\left(\frac{1}{6}\right) = \sin^{-1} 0.1667 = 9^\circ 36'
\]

\[
\cos \theta = \cos 9^\circ 36' = 0.9860
\]

The tension in the string,

\[
F = \frac{mg}{\cos \theta} = \frac{0.15 \times 9.8}{0.9860} = 1.491 \text{ N}
\]

(4) A pendulum, consisting of a light string of length 20 cm and a bob of mass 100 g, is set up as a conical pendulum. Its bob revolves at 75 rpm. Calculate the kinetic energy and the increase in gravitational potential energy of the bob. [Take \( \pi^2 = 10 \)] (4 marks)

Solution:

Data: \( L = 0.2 \text{ m}, m = 0.1 \text{ kg}, n = \frac{75}{60} = \frac{5}{4} \text{ rps}, g = 10 \text{ m/s}^2 \), \( \pi^2 = 10 \)

\[
\therefore T^2 = 4\pi^2 \frac{L \cos \theta}{g}
\]

\[
\therefore h = L \cos \theta = \frac{gT^2}{4\pi^2} = \frac{(10)(0.8)^2}{4(10)} = 0.16 \text{ m} \quad \ldots \ (1)
\]

\[
\therefore \cos \theta = \frac{0.16}{0.2} = 0.8
\]

\[
\therefore \theta = \cos^{-1} 0.8 = 36.87^\circ = 36^\circ 5'
\]

\[
v^2 = rg \tan \theta = (L \sin \theta) \ g \tan 36.87^\circ = (0.12) \ (10) \ (0.7500) = 0.9
\]

The KE of the bob \( \frac{1}{2} mv^2 = \frac{1}{2} (0.1)(0.9) = 0.045 \text{ J} \)

The increase in gravitational PE,

\[
\Delta PE = mg (L - h) = (0.1) \ (0.2 - 0.16) = 0.04 \text{ J}
\]

(5) A stone of mass 1 kg, attached at the end of a 1 m long string, is whirled in a horizontal circle. If the string makes an angle of 30° with the vertical, calculate the centripetal force acting on the stone. (2 marks)

Solution:

Data: \( m = 1 \text{ kg}, L = 1 \text{ m}, \theta = 30^\circ, g = 10 \text{ m/s}^2 \)
The centripetal force is
\[ \frac{mv^2}{r} = \frac{m(r\tan \theta)}{r} \]
\[ = mg \tan \theta \]
\[ = (10) (\tan 30^\circ) \]
\[ = 10 \times \frac{1}{\sqrt{3}} = \frac{10}{1.732} = 5.774 \text{ N} \]

\[ \text{Unit} \]

1.4 Vertical circular motion in Earth's gravity
1.4.1 Point mass undergoing vertical circular motion in Earth's gravity
1.4.2 Sphere (globe) of death
1.4.3 Vehicle on a convex bridge

Q. 57. What is vertical circular motion? Comment on its two types. (2 marks)
Ans. A body revolving in a vertical circle in the gravitational field of the Earth is said to perform vertical circular motion.

A vertical circular motion controlled only by gravity is a nonuniform circular motion because the linear speed of the body does not remain constant although the motion can be periodic.

In a controlled vertical circular motion, such as that a body attached to a rod, the linear speed of the body can be constant (including zero) so that such a motion can be uniform and periodic.

Q. 58. Using the principle of energy conservation, derive the expressions for the minimum speeds at different locations along a vertical circular motion controlled by gravity.
OR
Derive expressions for the linear speed at the lowest point, highest point and midway, of a particle revolving in a vertical circle if it has to just complete the circular motion without the string slackening at the top. (4 marks)
Ans. Consider a particle of mass \( m \) attached to a string and revolved in a vertical circle of radius \( r \). At every instant of its motion, the particle is acted upon by its weight \( mg \) and the tension \( T \) in the string. The particle may not complete the circle if the string slackens before the particle reaches the top. This requires that the particle must have some minimum speed.

(i) At the top (point A) : Let \( v_1 \) be the speed of the particle and \( T_1 \) the tension in the string. Here, both \( T_1 \) and weight \( mg \) are vertically downward. Hence, the net force on the particle towards the centre O is \( T_1 + mg \), which is the necessary centripetal force.
\[ \therefore T_1 + mg = \frac{mv_1^2}{r} \] \quad ... (1)

To find the minimum value of \( v_1 \) that the particle must have at the top, we consider the limiting case when the tension \( T_1 \) just becomes zero.
\[ \therefore \frac{mv_1^2}{r} = mg \]

that is, the particle’s weight alone is the necessary centripetal force at the point A.
\[ \therefore v_1 = \sqrt{gr} \] \quad ... (2)

(ii) At the bottom (point B) : Let \( v_2 \) be the speed at the bottom. Taking the reference level for zero potential energy to be the bottom of the circle, the particle has only kinetic energy \( \frac{1}{2}mv_2^2 \) at the lowest point.

Total energy at the bottom
\[ = \text{KE} + \text{PE} \]
\[ = \frac{1}{2}mv_2^2 + 0 = \frac{1}{2}mv_2^2 \] \quad ... (3)

As the particle goes from the bottom to the top of the circle, it rises through a height \( h = 2r \). Therefore, its potential energy at the top is
\[ mgh = mg(2r) \]
and, from Eq. (2), its minimum kinetic energy there is
\[ \frac{1}{2}mv_1^2 = \frac{1}{2}mgr \] \quad ... (4)

Minimum total energy at the top
\[ = \text{KE} + \text{PE} \]
Assuming that the total energy of the particle is conserved, total energy at the bottom = total energy at the top. Then, from Eqs. (4) and (5),

\[
\frac{1}{2} mv^2 = \frac{5}{2} mgr
\]

The minimum speed the particle must have at the lowest position is

\[
v_2 = \sqrt{5gr}
\]

(iii) At the midway (point C) : Let \( v_3 \) be the speed at point C, so that its kinetic energy is \( \frac{1}{2} mv_3^2 \).

At C, the particle is at a height \( r \) from the bottom of the circle. Therefore, its potential energy at C is \( mgr \).

Total energy at C

\[
= \frac{1}{2} mv_3^2 + mgr
\]

From the law of conservation of energy,

total energy at C = total energy at B

\[
\therefore \quad \frac{1}{2} mv_3^2 + mgr = \frac{5}{2} mgr
\]

\[
\therefore \quad v_3^2 = 5gr - 2gr = 3gr
\]

\[
\therefore \quad \text{The minimum speed the particle must have midway up is}
\]

\[
v_3 = \sqrt{3gr}
\]

Q. 59. Show that for looping a loop in a vertical plane, the critical velocity to cross over the highest point of the circle is \( \sqrt{gr} \). (2 marks)

Ans. Refer to the answer to Q. 58 and derive up to Eq. (2).

Q. 60. A body, tied to a string, performs circular motion in a vertical plane such that the tension in the string is zero at the highest point. What is the linear speed of the body at the (i) lowest position (ii) highest position? (1 mark)

Ans. (i) \( \sqrt{5rg} \) (ii) \( \sqrt{rg} \) in the usual notation.

Q. 61. A body, tied to a string, performs circular motion in a vertical plane such that the tension in the string is zero at the highest point. What is the angular speed of the body at the (i) highest position (ii) lowest position? (1 mark)

Ans. (i) \( \sqrt{g/r} \) (ii) \( \sqrt{5g/r} \) in the usual notation.

Q. 62. In a vertical circular motion, is zero speed possible at the top (uppermost point)? Under what condition(s)? (1 mark)

Ans. In a nonuniform vertical circular motion, e.g., those of a small body attached to a string or the loop-the-loop maneuvers of an aircraft or motorcycle or skateboard, the body must have some minimum speed to reach the top and complete the circle. In this case, the motion is controlled only by gravity and zero speed at the top is not possible.

However, in a controlled vertical circular motion, e.g., those of a small body attached to a rod or the giant wheel (Ferris wheel) ride, the body or the passenger seat can have zero speed at the top, i.e., the motion can be brought to a stop.

Q. 63. A small body, tied to a string and revolved in a vertical circle of radius \( r \). Prove that the difference in the tensions in the string at the highest and the lowest points is 6 times the weight of the body. OR

Derive an expression for the difference in tensions at the highest and lowest points for a particle performing vertical circular motion controlled by gravity. OR

In the vertical circular motion of a body controlled by gravity, prove that the difference between the extreme tensions (or normal forces) depends only upon the weight of the body. (3 marks)

Ans. Consider a small body (or particle) of mass \( m \) tied to a string and revolved in a vertical circle of radius \( r \) at a place where the acceleration due to gravity is \( g \). At every instant of its motion, the body is acted upon by two forces, namely, its weight \( mg \) and the tension \( T \) in the string.

Let \( v_2 \) be the speed of the body and \( T_2 \) be the tension in the string at the lowest point B. We take the reference level for zero potential energy to be the bottom of the circle. Then, the body has only kinetic energy \( \frac{1}{2} mv_2^2 \) at the lowest point.

\[
\therefore \quad T_2 = \frac{mv_2^2}{r} + mg
\]
and the total energy at the bottom = KE + PE

\[ KE = \frac{1}{2} mv_2^2 + 0 \]

\[ PE = \frac{1}{2} mv_2^2 \]  ... (2)

Let \( v_1 \) be the speed and \( T_1 \) the tension in the string at the highest point A. As the body goes from B to A, it rises through a height \( h = 2r \).

\[ T_1 = \frac{mv_1^2}{r} - mg \]  ... (3)

and the total energy at A = KE + PE

\[ KE = \frac{1}{2} mv_1^2 + mg (2r) \]  ... (4)

Then, from Eqs. (1) and (3),

\[ T_2 - T_1 = \frac{mv_2^2}{r} + mg - \left( \frac{mv_1^2}{r} - mg \right) \]

\[ = \frac{m}{r} (v_2^2 - v_1^2) + 2 mg \]  ... (5)

Assuming that the total energy of the body is conserved, the total energy at the bottom is equal to the energy at the top.

Then, from Eqs. (2) and (4),

\[ \frac{1}{2} mv_2^2 = \frac{1}{2} mv_1^2 + mg (2r) \]

\[ \therefore v_2^2 - v_1^2 = 4gr \]  ... (6)

Substituting this in Eq. (5),

\[ T_2 - T_1 = \frac{m}{r} (4gr) + 2 mg \]

\[ = 4mg + 2mg \]

\[ = 6mg \]

Therefore, the difference in the tensions in the string at the highest and the lowest points is 6 times the weight of the body.

**Q. 64. In a vertical circular motion controlled by gravity, derive an expression for the speed at an arbitrary position. Hence, show that the speed decreases while going up and increases while coming down.** (4 marks)

**OR**

In a nonuniform vertical circular motion, derive expressions for the speed and tension/normal force at an arbitrary position. (4 marks)

**OR**

Show that a vertical circular motion controlled by gravity is a nonuniform circular motion. (3 marks)

**Ans.** Consider a small body of mass \( m \) tied to a string and revolved in a vertical circle of radius \( r \). At every instant of its motion, the body is acted upon by its weight \( mg \) and the tension \( T \) in the string. At any instant, when the body is at the position P (Fig. 1.19), let the string make an angle \( \theta \) with the vertical. \( mg \) is resolved into components, \( mg \cos \theta \) (radial) and \( mg \sin \theta \) (tangential).

At point P shown, the net force on the body towards the centre, \( T - mg \cos \theta \), is the necessary centripetal force on the body. If \( v \) is its speed at P,

\[ T = \frac{mv^2}{r} + mg \cos \theta \]  ... (1)
Fig. 1.19 : Vertical circular motion

Let \( v_2 \) be the speed of the body at the lowest point B, which is the reference level for zero potential energy. Then, the body has only kinetic energy \( \frac{1}{2}mv_2^2 \) at B.

Total energy at B = KE + PE
\[
= \frac{1}{2}mv_2^2 + 0
\]
\[
= \frac{1}{2}mv_2^2
\] … (2)

As the body goes from B to P, it rises through a height \( h = r - r \cos \theta = r (1 - \cos \theta) \).

Total energy at P = KE + PE
\[
= \frac{1}{2}mv^2 + mgh
\]
\[
= \frac{1}{2}mv^2 + mgr (1 - \cos \theta)
\] … (3)

Assuming that the total energy of the body is conserved, total energy at any point = total energy at the bottom.

Then, from Eqs. (2) and (3),
\[
\frac{1}{2}mv^2 + mgr (1 - \cos \theta) = \frac{1}{2}mv_2^2
\]
\[
\therefore \quad v^2 = v_2^2 - 2gr (1 - \cos \theta)
\] … (4)
\[
\therefore \quad v = \sqrt{v_2^2 - 2gr (1 - \cos \theta)}
\] … (5)

From the above expression, it can be seen that the linear speed \( v \) changes with \( \theta \). Thus, as \( \theta \) increases, (while going up) \( \cos \theta \) decreases, \( 1 - \cos \theta \) increases, and \( v \) decreases. While coming down, \( \theta \) decreases and \( v \) increases. Hence, a vertical circular motion controlled by gravity is a nonuniform circular motion.

Substituting for \( v^2 \) from Eq. (4) in Eq. (1),
\[
T = \frac{m}{r} [v_1^2 - 2gr (1 - \cos \theta)] + mg \cos \theta
\]
\[
= \frac{mv_1^2}{r} - 2mg + 2mg \cos \theta + mg \cos \theta
\]
\[
\therefore \quad T = \frac{mv_1^2}{r} - mg (2 - 3 \cos \theta)
\] … (6)

Equation (6) is the expression for the tension in the string at any instant in terms of the speed at the lowest point.

Q. 65. A body at the end of a rod is revolved in a non-uniform vertical circular motion. Show that (i) it must have a minimum speed \( 2\sqrt{gr} \) at the bottom (ii) the difference in tensions in the rod at the highest and lowest positions is \( 6mg \). (4 marks)

Ans. Consider a body of mass \( m \) attached to a rod and revolved in a vertical circle of radius \( r \) at a place where the acceleration due to gravity is \( g \). We shall assume that the rod is not rigid so that the tension in the rod changes. As the rod is rotated in a non-uniform circular motion, the tension in the rod changes from a minimum value \( T_1 \) when the body is at the highest point to a maximum value \( T_2 \) when the body is at the bottom of the circle. At every instant, the body is acted upon by two forces, namely, its weight \( mg \) and the tension \( T \) in the string.

Let \( v_1 \) and \( v_2 \) be the speeds at the highest point A and lowest point B.
At the top, both \( \vec{T}_1 \) and weight \( \vec{mg} \) are vertically downward.
\[
\therefore \quad T_1 + mg = \frac{mv_1^2}{r}
\]
Taking the minimum value of \( v_1 = 0 \),
\[
T_1 + mg = 0
\] … (1)
Also, taking the reference level for zero potential energy
energy to be the bottom of the circle, the total energy at the top

\[ KE + PE = \frac{1}{2} mv_1^2 + mg(2r) \]

\[ = 2mgr \quad \therefore \ v_1 = 0 \quad \text{(2)} \]

At the bottom,

\[ T_2 - mg = \frac{mv_2^2}{r} \quad \text{(3)} \]

and the total energy = KE + PE

\[ = \frac{1}{2} mv_2^2 + 0 = \frac{1}{2} mv_2^2 \quad \text{(4)} \]

Assuming that the total energy of the body is conserved, from Eqs. (2) and (4),

\[ \frac{1}{2} mv_2^2 = 2mgr \]

\[ \therefore \ v_2^2 = 4gr \quad \therefore \ v_2 = 2\sqrt{gr} \quad \text{(5)} \]

Equation (5) gives the expression for the minimum speed at the bottom.

Substituting for \( v_2 \) in Eq. (3),

\[ T_2 - mg = \frac{m}{r} (4gr) = 4mg \]

\[ \therefore T_2 = 5mg \quad \text{(6)} \]

Therefore, from Eqs. (1) and (6), the difference in the tensions at the highest and lowest points of the circle is

\[ T_2 - (T_1 + mg) = 5mg \]

\[ \therefore T_2 - T_1 = 6mg \quad \text{(7)} \]

as required.

Q. 66. You may have seen in a circus a motorcyclist driving in vertical loops inside a hollow globe (sphere of death). Explain clearly why the motorcyclist does not fall down when at the highest point of the chamber. (2 marks)

Ans. A motorcyclist driving in vertical loops inside a hollow globe performs vertical circular motion. Suppose the mass of the motorcycle and motorcyclist is \( m \) and the radius of the chamber is \( r \). At every instant of the motion, the motorcyclist is acted upon by the weight \( mg \) and the normal reaction \( N \).

At the highest point, let \( v_1 \) be the speed and \( N_1 \) the normal reaction. Here, both \( N_1 \) and \( mg \) are parallel, vertically downward. Hence, the net force on the motorcyclist towards the centre \( O \) is \( N_1 + mg \). If this force is able to provide the necessary centripetal force at the highest point, the motorcycle does not lose contact with the globe and fall down. The minimum value of this force is found from the limiting case when \( N_1 \) just becomes zero and the weight alone provides the necessary centripetal force:

\[ \frac{mv_1^2}{r} = mg \]

This requires that the motorcycle has a minimum speed at the highest point given by

\[ v_1^2 = gr \quad \text{or} \quad v_1 = \sqrt{gr} \]

[Note: The ‘globe of death’ is a circus stunt in which stunt drivers ride motorcycles inside a mesh globe. Starting from small horizontal circles, they eventually perform revolutions along vertical circles. The linear speed is more for larger circles but angular speed is more for smaller circles as in conical pendulum.]

Do you know? (Textbook page 12)
A roller coaster is a common ride in amusement parks during which all aspects of vertical circular motion are encountered. The changing magnitude of the normal reaction the seat exerts on the passenger gives the feelings of weightlessness or accelerations of several \( g \)'s.

Use your brain power (Textbook page 12)
What is expected to happen if one travels fast over a speed breaker? Why?

The maximum speed with which a car can travel over a road surface, which is in the form of a convex arc of radius \( r \), is \( \sqrt{rg} \) where \( g \) is the acceleration due to gravity. For a speed breaker, \( r \) is very small (of the order of 1 m). Hence, one must slow down considerably while going over a speed breaker. Otherwise, the car will lose contact with the road and land with a thud.

How does the normal force on a concave suspension bridge change when a vehicle is travelling on it with a constant speed?

At the lowest point, \( N - mg \) provides the centripetal force. Therefore, \( N - mg = \frac{mv^2}{r} \), so that \( N = m \left( \frac{g + \frac{v^2}{r}}{r} \right) \).

Therefore, \( N \) increases with increasing \( v \).

Q. 67. A car crosses over a bridge which is in the form of a convex arc with a uniform speed. (i) State the expression for the normal reaction on the car. OR
How does the normal reaction on the car vary with speed? (1 mark)

(ii) Hence show that the maximum speed with which the car can cross the bridge without losing contact with the road is equal to \( \sqrt{rg} \). (1 mark)

Ans. Suppose a car of mass \( m \), travelling with a uniform speed \( v \), crosses over a bridge which is in the form of a convex arc of radius \( r \).

(i) The forces acting on it at the highest point are as shown in Fig. 1.21. Their resultant \( mg - N \) provides the centripetal force.

\[ \therefore \ mg - N = \frac{mv^2}{r} \]

\[ \therefore \ N = m \left( g - \frac{v^2}{r} \right) \]  

... (1)

is the required expression. It shows that as \( v \) increases, \( N \) decreases.

(ii) Equation (1) shows that for \( g - \frac{v^2}{r} = 0 \), i.e., for centripetal acceleration equalling the gravitational acceleration, \( N = 0 \). That is, for \( \frac{v^2}{r} = g \) or \( v = \sqrt{rg} \), the car just loses contact with the road. Therefore, this is the maximum speed with which a car can cross the bridge, irrespective of its mass.

**Solved Problems 1.4 – 1.4.3**

[Data : Take \( g = 10 \text{ m/s}^2 \) unless specified otherwise]

Q. 68. Solve the following:

(1) An object of mass 1 kg tied to one end of a string of length 9 m is whirled in a vertical circle. What is the minimum speed required at the lowest position to complete the circle? [\( g = 9.8 \text{ m/s}^2 \)] (2 marks)

Solution:

Data : \( m = 1 \text{ kg}, \ r = 9 \text{ m}, \ g = 9.8 \text{ m/s}^2 \)

The minimum speed of the object at the lowest position is

\[ v = \sqrt{rg} = \sqrt{9 \times 9.8} = \sqrt{9 	imes 49} = 21 \text{ m/s} \]

(2) A stone of mass 5 kg, tied at one end of a rope of length 0.8 m, is whirled in a vertical circle. Find the minimum velocity at the highest point and at the midway point. [\( g = 9.8 \text{ m/s}^2 \)] (3 marks)

Solution:

Data : \( m = 5 \text{ kg}, \ r = 0.8 \text{ m}, \ g = 9.8 \text{ m/s}^2 \)

(i) The minimum velocity of the stone at the highest point in its path,

\[ v = \sqrt{rg} = \sqrt{0.8 \times 9.8} = 2.8 \text{ m/s} \]

(ii) The minimum velocity of the stone at the midway point in its path,

\[ v = \sqrt{3rg} = \sqrt{3 \times 0.8 \times 9.8} = 4.85 \text{ m/s} \]

(3) A small body of mass 0.3 kg oscillates in a vertical plane with the help of a string 0.5 m long with a constant speed of 2 m/s. It makes an angle of 60° with the vertical. Calculate the tension in the string. (2 marks)

Solution:

Data : \( m = 0.3 \text{ kg}, \ r = 0.5 \text{ m}, \ v = 2 \text{ m/s}, \ \theta = 60^\circ, \ g = 10 \text{ m/s}^2 \)

\[ \frac{mv^2}{r} = T - mg \cos \theta \]

Tension in the string,

\[ T = \frac{mv^2}{r} + mg \cos \theta \]

\[ = \frac{(0.3)(2)^2}{0.5} + (0.3)(10) \cos 60^\circ \]

\[ = 2.4 + 0.3 \times 10 \times \frac{1}{2} = 2.4 + 0.3 \times 5 = 3.9 \text{ N} \]

(4) A bucket of water is whirled in a vertical circle at an arm’s length. Find the minimum speed at the top so that no water spills out. Also find the corresponding angular speed. [Assume \( r = 0.75 \text{ m} \)] (3 marks)

Solution:

Data : \( r = 0.75 \text{ m}, \ g = 10 \text{ m/s}^2 \)

At the highest point the minimum speed required is

\[ v = \sqrt{rg} = \sqrt{0.75 \times 10} = 2.738 \text{ m/s} \]
The corresponding angular speed is

$$\omega = \frac{v}{r} = \frac{2.738}{0.75} = 3.651 \text{ rad/s}$$

(5) A pendulum, with a bob of mass $m$ and string length $l$, is held in the horizontal position and then released into a vertical circle. Show that at the lowest position the velocity of the bob is $\sqrt{2gl}$ and the tension in the string is $3mg$. (3 marks)

**Solution:**

Taking the reference level for zero potential energy to be the bottom of the vertical circle, the initial potential energy of the bob at the horizontal position is $mgh = mg \ell$.

Hence, at the bottom where the speed of the bob is $v$, it has only kinetic energy $\frac{1}{2}mv^2$.

By the principle of conservation of energy,

$$\frac{1}{2}mv^2 = mgh$$

$$\therefore \quad v^2 = 2gl$$

$$\therefore \quad v = \sqrt{2gl} \quad \ldots \quad (1)$$

This gives the required velocity at the lowest position.

Also, at the bottom, the tension ($T$) and the centripetal acceleration are upward while the force of gravity is downward.

$$\therefore \quad T - mg = \frac{mv^2}{r}$$

$$\therefore \quad T = \frac{mv^2}{r} + mg$$

$$= \frac{m}{I} (2gl) + mg = 2mg + mg$$

$$= 3mg \quad \ldots \quad (2)$$

Equations (1) and (2) give the required expressions for the velocity and tension at the lowest position.

(6) A stone of mass 100 g attached to a string of length 50 cm is whirled in a vertical circle by giving it a velocity of 7 m/s at the lowest point. Find the velocity at the highest point. (3 marks)

**Solution:**

Data : $m = 0.1 \text{ kg}$, $r = l = 0.5 \text{ m}$, $v_2 = 7 \text{ m/s}$,

$g = 10 \text{ m/s}^2$

The total energy at the bottom, $E_{bot}$

$$= KE + PE = \frac{1}{2} mv_2^2 + 0 = \frac{1}{2} (0.1) (7)^2 = 2.45 \text{ J}$$

The total energy at the top, $E_{top}$

$$= KE + PE = \frac{1}{2} mv_1^2 + mg (2r)$$

$$= \frac{1}{2} (0.1) v_1^2 + (0.1) (10) (2 \times 0.5)$$

$$= 0.05 v_1^2 + 1$$

By the principle of conservation of energy,

$$E_{top} = E_{bot}$$

$$\therefore \quad 0.05 v_1^2 + 1 = 2.45$$

$$\therefore \quad v_1^2 = \frac{2.45 - 1}{0.05} = \frac{145}{5} = 29$$

$$\therefore \quad v_1 = \sqrt{29} = 5.385 \text{ m/s}$$

(7) A pilot of mass 50 kg in a jet aircraft executes a “loop-the-loop” manoeuvre at a constant speed of 250 m/s. If the radius of the vertical circle is 5 km, compute the force exerted by the seat on the pilot at (i) the top of the loop (ii) the bottom of the loop. (3 marks)

**Solution:**

Data : $m = 50 \text{ kg}$, $v = 250 \text{ m/s}$, $r = 5 \text{ km} = 5 \times 10^3 \text{ m}$,

$g = 10 \text{ m/s}^2$
(i) At the top of the loop: The forces on the pilot are the gravitational force \( mg \) and the normal force \( N_1 \), exerted by the seat, both acting downward. So the net force downward that causes the centripetal acceleration has a magnitude \( N_1 + mg \).

\[
\therefore \quad N_1 + mg = \frac{mv^2}{r} \\
\therefore \quad N_1 = m \left( \frac{v^2}{r} - g \right)
\]

\[
= 50 \left[ \frac{(250)^2}{5 \times 10^3} - 9.8 \right] = 50 (12.5 - 10) \\
= 50(2.5) = 125 \text{ N}
\]

(ii) At the bottom of the loop: The forces on the pilot are the downward gravitational force \( mg \) and the upward normal force \( N_2 \), exerted by the seat. So the net upward force that causes the centripetal acceleration has a magnitude \( N_2 - mg \).

\[
\therefore \quad N_2 - mg = \frac{mv^2}{r} \\
\therefore \quad N_2 = m \left( \frac{v^2}{r} + g \right)
\]

\[
= 50 \left[ \frac{(250)^2}{5 \times 10^3} + 10 \right] = 50 (12.5 + 10) \\
= 50 (22.5) = 1125 \text{ N}
\]

The forces exerted by the seat on the pilot at the top and bottom of the loop are 125 N and 1125 N, respectively.

(8) A ball released from a height \( h \) along an incline, slides along a circular track of radius \( R \) (at the end of the incline) without falling vertically downwards. Show that \( h_{\text{min}} = \frac{5}{2} R \). (2 marks)

Solution:

To just loop-the-loop, the ball must have a speed \( v_2 = \sqrt{5Rg} \) at the bottom of the circular track.

If \( h_{\text{min}} \) is the minimum height above the bottom of the circular track from which the ball must be released, by the principle of conservation of energy, we have,

\[
mgh_{\text{min}} = \frac{1}{2} mv_2^2 = \frac{1}{2} m (5Rg)
\]

\[
\therefore \quad h_{\text{min}} = \frac{5}{2} R = 2.5R
\]

(9) A block of mass 1 kg is released from P on a frictionless track which ends with a vertical quarter circular turn (Fig. 1.25).

What are the magnitudes of the radial acceleration and total acceleration of the block when it arrives at Q? (3 marks)

Solution:

Data: \( m = 1 \text{ kg}, \ h = 6 \text{ m}, \ r = 2 \text{ m}, \ g = 10 \text{ m/s}^2 \)

Let \( v \) be the speed of the block at Q. Then, the total energy of the block at Q is

\[
E = KE + PE = \frac{1}{2} mv^2 + mgr
\]
By the principle of conservation of energy,

\[ mgh = \frac{1}{2}mv^2 + mgr \]

\[ \therefore v^2 = 2g(h - r) \]

The magnitude of the radial acceleration of the block at Q is

\[ a_c = \frac{v^2}{r} = \frac{2g(h - r)}{r} = \frac{2 \times 10(6 - 2)}{2} = 40 \text{ m/s}^2 \]

The magnitude of the total acceleration of the block at Q is

\[ a = \sqrt{a_c^2 + g^2} = \sqrt{(40)^2 + (10)^2} = \sqrt{1600 + 100} = \sqrt{1700} = 41.23 \text{ m/s}^2 \]

Let \( \alpha \) be the angle between \( \vec{a}_c \) (horizontal) and \( \vec{a} \). Then,

\[ \alpha = \tan^{-1} \frac{g}{a_c} = \tan^{-1} \frac{10}{40} = \tan^{-1} 0.25 = 14^\circ 2' \]

The radial acceleration has a magnitude 40 m/s\(^2\). The total acceleration has a magnitude 41.23 m/s\(^2\) and makes an angle of 14\(^\circ 2'\) with the radial acceleration.

(10) A loop-the-loop cart runs down an incline into a vertical circular track of radius 3 m and then describes a complete circle. Find the minimum height above the top of the circular track from which the cart must be released. \( (2 \text{ marks}) \)

Solution:

By the principle of conservation of energy, we have,

\[ mgh = \frac{1}{2}mv^2 + mgr \]

\[ \therefore h = \frac{3}{2} = 1.5 \text{ m} \]

*(11)* A motorcyclist (treated as a particle) is undergoing vertical circles inside a sphere of death. The speed of the motorcycle varies between 6 m/s and 10 m/s. Calculate diameter of the sphere of death. How much minimum values are possible for these two speeds? \( (3 \text{ marks}) \)

Solution:

Data:

- \( v_{\text{top}} = 6 \text{ m/s} \)
- \( v_{\text{bot}} = 10 \text{ m/s} \)
- \( g = 10 \text{ m/s}^2 \)

\[ v_{\text{top}}^2 = v_{\text{bot}}^2 + 4gr \]

\[ \therefore r = \frac{v_{\text{bot}}^2 - v_{\text{top}}^2}{4g} = \frac{(10)^2 - (6)^2}{4 \times 10} = \frac{64}{40} = 1.6 \text{ m} \]

The diameter of the sphere of death = 3.2 m.

For this \( r \), \( v_{\text{min}} = \sqrt{gr} \) at the top.

\[ \therefore v_{\text{min}} = \sqrt{10 \times 1.6} = \sqrt{16} = 4 \text{ m/s} \]

The corresponding minimum speed at the bottom

\[ = \sqrt{5gr} = \sqrt{5(10)(1.6)} = \sqrt{80} = 4\sqrt{5} \text{ m/s} \]

The required minimum values of the speeds are 4 m/s and \( 4\sqrt{5} \) m/s.

(12) A motorcyclist rides in vertical circles in a hollow sphere of radius 5 m. Find the required minimum speed and minimum angular speed, so that he does not lose contact with the sphere at the highest point. \( [g = 9.8 \text{ m/s}^2] \) \( (3 \text{ marks}) \)

Solution:

Data:

- \( r = 5 \text{ m} \)
- \( g = 9.8 \text{ m/s}^2 \)

Let \( v \) and \( \omega \) be respectively the required minimum speed and angular speed at the highest point.

(i) \[ v = \sqrt{rg} = \sqrt{5 \times 9.8} = \sqrt{49} = 7 \text{ m/s} \]

(ii) \[ \omega = \frac{v}{r} = \frac{7}{5} = 1.4 \text{ rad/s} \]

\[ \omega = \frac{\sqrt{g}}{r} = \frac{9.8}{5} = \sqrt{1.96} = 1.4 \text{ rad/s} \]

(13) The vertical section of a road over a bridge in the direction of its length is in the form of an arc of a circle of radius 4.4 m. Find the maximum speed with which a vehicle can cross the bridge without
(14) A small body tied to a string is revolved in a vertical circle of radius \( r \) such that its speed at the top of the circle is \( \sqrt{2gr} \). Find
(i) the angular position of the string when the tension in the string is numerically equal to 5 times the weight of the body.  
(ii) the KE of the body at this position (2 marks)
(iii) the minimum and maximum KEs of the body. (2 marks)

[Take \( m = 0.1 \text{ kg} \), \( r = 1.2 \text{ m} \), \( g = 10 \text{ m/s}^2 \)]

Solution :

Data : \( v_{\text{top}} = \sqrt{2gr} \), \( T = 5mg \), \( m = 0.1 \text{ kg} \), \( r = 1.2 \text{ m} \), \( g = 10 \text{ m/s}^2 \)

Let the angular position of the string, \( \theta = 0^\circ \) when the body is at the bottom of the circle.

We assume total energy to be conserved and take the reference level for zero potential energy to be the bottom of the circle.

Total energy at the top, \( E \)
\[
E = KE + PE = \frac{1}{2}mv_{\text{top}}^2 + mg (2r)
\]
\[
= \frac{1}{2}m (2gr) + 2mgr = 3mgr
\]

(i) At position P where \( T = 5mg \),
\[
T - mg \cos \theta = \frac{mv^2}{r}
\]
\[
\therefore \ 5mg - mg \cos \theta = \frac{mv^2}{r}
\]
\[
\therefore \ v^2 = gr (5 - \cos \theta) \quad \ldots \ (2)
\]

At P, the vertical displacement of the body from the bottom is \( r(1 - \cos \theta) \). Its total energy is also \( E \).
\[
\therefore \ KE + PE = E
\]
\[
\therefore \ \frac{1}{2}mv^2 + mgr (1 - \cos \theta) = 3mgr \quad \ldots \ [\text{from Eq. (1)}]
\]

Substituting for \( v^2 \) from Eq. (2) and simplifying
\[
gr (5 - \cos \theta) + 2gr (1 - \cos \theta) = 6gr
\]
\[
\therefore \ 5 - \cos \theta + 2 - 2 \cos \theta = 6
\]
\[
\therefore \ 3 \cos \theta = 1 \quad \therefore \ \cos \theta = \frac{1}{3}
\]
\[
\therefore \ \theta = \cos^{-1}(0.3333) = 70^\circ 3'
\]

(ii) At P, \( v^2 = gr (5 - \cos \theta) \)
\[
= (10)(1.2)(5 - \frac{1}{3}) = 12 \times \frac{14}{3}
\]
\[
= 56 \text{ (m/s)}^2
\]

The KE of the body
\[
= \frac{1}{2}mv^2 = \frac{1}{2}(0.1)(56) = 2.8 \text{ J}
\]

(iii) The body has minimum KE at the top and maximum KE at the bottom.
\[
\therefore \ KE_{\text{min}} = \frac{1}{2}mv^2_{\text{top}} = \frac{1}{2}(0.1)(2gr)
\]
\[
= (0.1)(10)(1.2) = 1.2 \text{ J}
\]
\[
v^2_{\text{bot}} = v^2_{\text{top}} + 4gr = 2gr + 4gr = 6gr
\]
\[
\therefore \ KE_{\text{max}} = \frac{1}{2}mv^2_{\text{bot}} = \frac{1}{2}m(6gr) = 3mgr
\]
\[
= 3(0.1)(10)(1.2) = 3.6 \text{ J}
\]

(15) An object of mass 0.5 kg attached to a rod of length 0.5 m is whirled in a vertical circle at a constant angular speed. If the maximum tension in the rod is 5 kg wt, calculate the linear speed of the object and the maximum number of revolutions it can complete in a minute. (3 marks)

Solution :

Data : \( m = 0.5 \text{ kg} \), \( r = l = 0.5 \text{ m} \), \( g = 10 \text{ m/s}^2 \), \( T_2 = 5 \text{ kg wt} = 5 \times 10 \text{ N} \)

As the rod is rotated in a vertical circle at a constant angular speed, the linear speed of the object at the end of the rod is constant, say \( v \). However, the tension in the rod changes from a minimum value \( T_1 \) when the object is at the highest point to a maximum value \( T_2 \) when the object is at the bottom of the circle.
At the bottom of the circle, the tension and acceleration are upward while the force of gravity is downward.

\[ T - mg = \frac{mv^2}{r} \]

\[ v^2 = r \left( \frac{T - g}{m} \right) \]

\[ = 0.5 \left( \frac{5 \times 10}{0.5} - 10 \right) \]

\[ = 50 - 5 = 45 \]

\[ v = \sqrt{45} = 6.708 \text{ m/s} \]

The period \( T \) of the motion is

\[ T = \frac{\text{circumference}}{\text{speed}} = \frac{2\pi r}{v} \]

\[ = \frac{2 \times 3.142 \times 0.5}{6.702} = 0.4688 \text{ s} \]

\[ v = \sqrt{r^2 g} \]

\[ = 0.1 \times (4 + 8.486) = 1.2486 \text{ N} \]

(17) A bucket of water is tied to one end of a rope 8 m long and rotated about the other end in a vertical circle. Find the number of revolutions per minute such that water does not spill. (2 marks)

Solution:

[Important note: The circular motion of the bucket in a vertical plane under gravity is not a uniform circular motion. Assuming the critical case of the motion such that the bucket has the minimum speed at the highest point required for the water to stay put in the bucket, we can find the minimum frequency of revolution.]

Data: \( r = 8 \text{ m}, \ g = 9.8 \text{ m/s}^2, \ \pi = 3.142 \)

Assuming the bucket has a minimum speed \( v = \sqrt{rg} \) at the highest point, the corresponding angular speed is

\[ \omega = \frac{v}{r} = \sqrt{\frac{g}{r}} \]

\[ = \frac{1}{2 \times 3.142} \sqrt{ \frac{9.8}{8} } \]

\[ = \frac{1}{6.284} \sqrt{1.225} \approx 0.1761 \text{ rps} \]

\[ = 0.1761 \times 60 \text{ rpm} = 10.566 \text{ rpm} \]

(16) A small body of mass \( m = 0.1 \text{ kg} \) at the end of a cord 1 m long swings in a vertical circle. Its speed is 2 m/s when the cord makes an angle \( \theta = 30^\circ \) with the vertical. Find the tension in the cord. (2 marks)

Solution:

Data: \( m = 0.1 \text{ kg}, \ r = 1 \text{ m}, \ v = 2 \text{ m/s}, \ \theta = 30^\circ, \ g = 9.8 \text{ m/s}^2 \)

When the cord makes an angle \( \theta \) with the vertical, the centripetal force on the body is

\[ \frac{mv^2}{r} = T - mg \cos \theta \]

The tension in the cord,

\[ T = \frac{mv^2}{r} + mg \cos \theta \]

\[ = 0.1 \left( \frac{2^2}{1} + 9.8 \times \cos 30^\circ \right) \]

\[ = 0.1 \left( 4 + 9.8 \times \frac{\sqrt{3}}{2} \right) = 0.1(4 + 4.9 \times 1.732) \]

\[ = 0.1(4 + 8.486) = 1.2486 \text{ N} \]

Q. 69. Derive an expression for the kinetic energy of a body rotating with constant angular velocity. (3 marks)

Ans. Consider a rigid body rotating with a constant angular velocity \( \vec{\omega} \) about an axis passing through the point O and perpendicular to the plane of the figure. Suppose that the body is made up of \( N \) particles of masses \( m_1, m_2, \ldots, m_N \) situated at perpendicular distances \( r_1, r_2, \ldots, r_N \), respectively, from the axis of rotation as shown in Fig. 1.27.

As the body rotates, all the particles perform uniform circular motion with the same angular velocity \( \vec{\omega} \). However, they have different linear speeds depending upon their distances from the axis of rotation.
The linear speed of the particle with mass \( m_1 \) is \( v_1 = r_1 \omega \). Therefore, its kinetic energy is

\[
E_1 = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 r_1^2 \omega^2 \quad \ldots \ (1)
\]

Similarly, the kinetic energy of the particle with mass \( m_2 \) is \( E_2 = \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_2 r_2^2 \omega^2 \), and so on.

The rotational kinetic energy of the body is

\[
E_{\text{rot}} = E_1 + E_2 + \ldots + E_N
\]

\[
= \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \ldots + \frac{1}{2} m_N r_N^2 \omega^2
\]

\[
= \frac{1}{2} \left( \sum_{i=1}^{N} m_i r_i^2 \right) \omega^2
\]

\[
\therefore \ E_{\text{rot}} = \frac{1}{2} I \omega^2 \quad \ldots \ (2)
\]

where \( I = \sum_{i=1}^{N} m_i r_i^2 \) is the moment of inertia of the body about the axis of rotation.

Equation (2) gives the required expression.

Q. 70. Derive an expression for the rotational kinetic energy of a body. State how it depends on the moment of inertia, and frequency and period of rotation. (4 marks)

Ans. Refer to the answer to Q. 69 and continue:

If \( f \) is the frequency of rotation of the body and \( T \) the period of rotation, then

\[
\omega = 2\pi f = \frac{2\pi}{T}
\]

\[
\therefore \ E_{\text{rot}} = \frac{1}{2} I (2\pi f)^2 = 2\pi^2 I f^2 = 2\pi^2 \frac{I}{T^2} \quad \ldots \ (3)
\]

Thus, from Eqs. (2) and (3), the rotational kinetic energy of a body is proportional to (1) its moment of inertia about the given rotation axis (2) the square of its angular speed (or frequency of rotation) or inverse square of its rotational period.

Q. 71. Define moment of inertia. State the factors which it depends on. Obtain its dimensions and state its SI unit. (3 marks) OR

Define moment of inertia. State its dimensions and SI units. (2 marks)

Ans.

(1) **Moment of inertia**: The moment of inertia of a body about a given axis of rotation is defined as the sum of the products of the masses of the particles of the body and the squares of their respective distances from the axis of rotation.

If the body is made up of \( N \) discrete particles of masses \( m_1, m_2, \ldots, m_N \) situated at respective distances \( r_1, r_2, \ldots, r_N \) from the axis of rotation, the moment of inertia of the body is

\[
I = m_1 r_1^2 + m_2 r_2^2 + \ldots + m_N r_N^2
\]

\[
= \sum_{i=1}^{N} m_i r_i^2 \quad \ldots \ (1)
\]

For a rigid body, having a continuous and uniform distribution of mass, the moment of inertia is

\[
I = \int r^2 dm
\]

where \( dm \) is the mass of an infinitesimal element, situated at distance \( r \) from the axis of rotation.

(2) The moment of inertia of a rigid body depends on (i) the mass and shape of the body (ii) orientation and position of the rotation axis (iii) distribution of the mass about the rotation axis.

(3) **Dimensions**:

\[
[\text{Moment of inertia}] = [\text{mass}] [\text{distance}]^2 = [M] [L^2] = [M^1 L^2 T^0]
\]

(4) **SI unit**: The kilogram-metre\(^2\) (kg·m\(^2\)).

[Note: The integral \( \int r^2 dm \) can be worked out for the moment of inertia of symmetrical, i.e., regularly shaped, bodies of continuous mass distribution in terms of their dimensions. For a body of irregular shape, however, the integral cannot be evaluated and its moment of inertia about a given axis must be found experimentally (from \( I = \tau/\omega \), see Q. 117).]
Q. 72. Explain the physical significance of moment of inertia. (2 marks)

Ans.

(1) The physical significance of moment of inertia can be understood by comparing the formulae in the following table.

<table>
<thead>
<tr>
<th>Linear motion</th>
<th>Rotational motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Momentum</td>
<td>1. Angular momentum</td>
</tr>
<tr>
<td>= mass × velocity</td>
<td>= moment of inertia × angular velocity</td>
</tr>
<tr>
<td>2. Force</td>
<td>2. Torque</td>
</tr>
<tr>
<td>= mass × acceleration</td>
<td>= moment of inertia × angular acceleration</td>
</tr>
<tr>
<td>3. Kinetic energy = (\frac{1}{2}Mv^2)</td>
<td>3. Kinetic energy = (\frac{1}{2}I\omega^2)</td>
</tr>
</tbody>
</table>

(2) Force produces acceleration, while torque produces angular acceleration. Force and torque are analogous quantities. Also, momentum and angular momentum are analogous quantities.

(3) By comparing the above formulae, we find that moment of inertia plays the same role in rotational motion as that played by mass in linear motion. The moment of inertia of a body is its rotational inertia, that which opposes any tendency to change its angular velocity. In the absence of a net torque, the body continues to rotate with a uniform angular velocity.

[Note: The relations for angular momentum and torque in rotational motion are given in Q. 113 and Q. 117, respectively.]

Q. 73. Three point masses \(M_1\), \(M_2\), and \(M_3\) are located at the vertices of an equilateral triangle of side \(a\). What is the moment of inertia of the system about an axis along the altitude of the triangle passing through \(M_1\)? (1 mark)

Ans.

The moment of inertia of the system about the altitude passing through \(M_1\) is

\[ I = M_1r_1^2 + M_2r_2^2 + M_3r_3^2 \]

Since \(M_1\) lies on the axis of rotation, \(r_1 = 0\).

Also, \(r_2 = r_3 = \frac{a}{2}\).

\[ \therefore \quad I = (M_2 + M_3) \left( \frac{a}{2} \right)^2 = \frac{1}{4} (M_2 + M_3)a^2 \]

Q. 74. Find the moment of inertia of a hydrogen molecule about its centre of mass if the mass of each hydrogen atom is \(m\) and the distance between them is \(R\). (1 mark)

Ans. We assume the rotation axis to be a transverse axis through the centre of mass of the linear molecule \(\text{H}_2\). Then, each of the hydrogen atom is a distance \(\frac{1}{2}R\) from the CM. Therefore, the MI of the molecule about this axis, \(I = 2m \left( \frac{R}{2} \right)^2 = \frac{1}{2} mR^2\).

[Notes: (1) For a \(\text{H}_2\) molecule, \(m_H = 1.674 \times 10^{-27}\) kg and bond length = \(7.774 \times 10^{-10}\) m, so that \(I = 5.065 \times 10^{-48}\) kg\cdot m\(^2\). (2) As atoms are treated as particles, we do not consider rotation about the line passing through the atoms.]

Solved Problems 1.5

Q. 75. Solve the following:

(1) Four particles of masses 0.2 kg, 0.3 kg, 0.4 kg and 0.5 kg respectively are kept at corners A, B, C and D of a square ABCD of side 1 m. Find the moment of inertia of the system about an axis passing through point A and perpendicular to the plane of the square. (2 marks)

Solution:

Data: \(m_1 = 0.2\) kg, \(m_2 = 0.3\) kg, \(m_3 = 0.4\) kg, \(m_4 = 0.5\) kg

Fig. 1.29
The axis of rotation passes through point A and is perpendicular to the plane of the square. Hence the distance \( r_1 \) of mass \( m_1 \) from the axis is \( r_1 = 0 \), that of mass \( m_2 \) is \( r_2 = AB = 1 \) m, that of mass \( m_3 \) is \( r_3 = AC = \sqrt{2} \) m and that of mass \( m_4 \) is \( r_4 = AD = 1 \) m.

The moment of inertia,

\[
I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + m_4 r_4^2
\]

\[
= 0.2 (0)^2 + 0.3 (1)^2 + 0.4 (\sqrt{2})^2 + 0.5 (1)^2
\]

\[= 1.6 \text{ kg} \cdot \text{m}^2\]

(2) The moment of inertia of the Earth about its axis of rotation is \( 9.83 \times 10^{37} \) kg·m² and its angular speed is \( 7.27 \times 10^{-5} \) rad/s. Calculate its (i) kinetic energy of rotation (ii) radius of gyration. [Mass of the Earth = \( 6 \times 10^{24} \) kg] (3 marks)

Solution:

Data : \( I = 9.83 \times 10^{37} \) kg·m², \( \omega = 7.27 \times 10^{-5} \) rad/s, \( M = 6 \times 10^{24} \) kg

(i) The kinetic energy of rotation of the Earth,

\[
E_{\text{rot}} = \frac{1}{2} I \omega^2 = \frac{1}{2} \times 9.83 \times 10^{37} \times (7.27 \times 10^{-5})^2
\]

\[= 2.598 \times 10^{29} \text{ J}\]

(ii) \( I = M k^2 \)

The radius of gyration, \( k = \sqrt{\frac{I}{M}} = \sqrt{\frac{9.83 \times 10^{37}}{6 \times 10^{24}}}
\]

\[= 4.048 \times 10^6 \text{ m}\]

Q. 76. State an expression for the moment of inertia of a thin ring about its transverse symmetry axis. (\( \frac{1}{2} \) mark)

Ans. A thin uniform ring (or hoop) has all its mass uniformly distributed along the circumference of a circle. It is taken to be a two-dimensional body. It is also assumed that the radial thickness of the ring is so small as to be completely negligible in comparison to its radius.
Since the disc is uniform, the area and mass of this elemental ring are

\[ A = 2\pi r \, dr \text{ and } dm = 2\pi\sigma r \, dr \]

\[ \therefore \sigma = \frac{dm}{A} = \frac{dm}{2\pi r \, dr} \] ... (2)

and its moment of inertia (MI) about the given axis is \( dm \cdot r^2 \).

Therefore, the MI of the disc is

\[ I = \int_0^R r^2 \, dm = \int_0^R r^2 (2\pi\sigma r \, dr) = 2\pi \int_0^R r^4 \, dr \] ... (3)

\[ \therefore I = 2\pi\sigma \frac{R^4}{4} = 2\pi \frac{M}{\pi R^2} \left( \frac{R^4}{4} \right) = \frac{1}{2} MR^2 \] ... (4)

This gives the required expression.

Q. 78. Define radius of gyration. Explain its physical significance. (2 marks) OR

Discuss the necessity of radius of gyration. Define it. On what factors does it depend and does not depend? (2 marks) OR

Why is it useful to define radius of gyration? (1 mark)

Ans. Definition: The radius of gyration of a body rotating about an axis is defined as the distance between the axis of rotation and the point at which the entire mass of the body can be supposed to be concentrated so as to give the same moment of inertia as that of the body about the given axis.

![Fig. 1.32: Concept of radius of gyration](image)

The moment of inertia (MI) of a body about a given rotation axis depends upon (i) the mass of the body and (ii) the distribution of mass about the axis of rotation. These two factors can be separated by expressing the MI as the product of the mass \( (M) \) and the square of a particular distance \( (k) \) from the axis of rotation. This distance is called the radius of gyration and is defined as given above. Thus,

\[ I = \sum_i m_i r_i^2 = Mk^2 \]

\[ \therefore k = \sqrt{\frac{I}{M}} \]

Physical significance: The radius of gyration is less if \( I \) is less, i.e., if the mass is distributed close to the axis; and it is more if \( I \) is more, i.e., if the mass is distributed away from the axis. Thus, it gives the idea about the distribution of mass about the axis of rotation.

Q. 79. Is radius of gyration of a rigid body a constant quantity? (1 mark)

Ans. Radius of gyration of a rigid body depends on the distribution of mass of the body about a rotation axis and, therefore, changes with the choice of the rotation axis. Hence, unlike the mass of the body which is constant, radius of gyration and moment of inertia of the body are not constant.

Q. 80. Can you find some similarity between the centre of mass and radius of gyration? (2 marks)

Ans. The centre of mass (CM) coordinates locates a point where the entire mass \( M \) of a system of particles or that of a rigid body can be thought to be concentrated such that the acceleration of this point mass obeys Newton’s second law of motion, viz., \( \vec{F}_{\text{net}} = M\vec{a}_{\text{CM}} \), where \( \vec{F}_{\text{net}} \) is the sum of all the external forces acting on the body or on the individual particles of the system of particles.

Similarly, radius of gyration locates a point from the axis of rotation where the entire mass \( M \) can be thought to be concentrated such that the angular acceleration of that point mass about the axis of rotation obeys the relation, \( \vec{\tau}_{\text{net}} = M\vec{\omega} \), where \( \vec{\tau}_{\text{net}} \) is the sum of all the external torques acting on the body or on the individual particles of the system of particles.

Q. 81. State an expression for the radius of gyration of (i) a thin ring (ii) a thin disc, about respective transverse symmetry axis. OR

Show that for rotation about respective transverse symmetry axis, the radius of gyration of a thin disc is less than that of a thin ring. (2 marks)
 Ans.
(i) The MI of the ring about the transverse symmetry axis is
\[ I_{CM} = MR^2 \]  … (1)

**Radius of gyration:** The radius of gyration of the ring about the transverse symmetry axis is
\[ k = \sqrt{\frac{I_{CM}}{M}} = \sqrt{R^2} = R \]  … (2)

(ii) The MI of the disc about the transverse symmetry axis is
\[ I_{CM} = \frac{1}{2} MR^2 \]  … (3)

**Radius of gyration:** The radius of gyration of the disc for the given rotation axis is
\[ k = \sqrt{\frac{I_{CM}}{M}} = \frac{\sqrt{M_d R_d^2 / 2}}{M_d} = \frac{1}{\sqrt{2}} R_d \]  … (4)

Therefore, \( k_{\text{disc}} < k_{\text{ring}} \).

\[ \star \text{Q. 82. What can you infer if a uniform ring and a uniform disc have the same radius of gyration?} \]

(2 marks)

**Ans.** The radius of gyration of a thin ring of radius \( R_r \) about its transverse symmetry axis is
\[ k_r = \sqrt{\frac{I_{CM}}{M_r}} = \sqrt{R_r^2} = R_r \]

The radius of gyration of a thin disc of radius \( R_d \) about its transverse symmetry axis is
\[ k_d = \sqrt{\frac{I_{CM}}{M_d}} = \frac{\sqrt{M_d R_d^2 / 2}}{M_d} = \frac{1}{\sqrt{2}} R_d \]

Given \( k_r = k_d \),
\[ R_r = \frac{1}{\sqrt{2}} R_d \] or, equivalently, \( R_d = \sqrt{2} R_r \).

\[ \star \text{Q. 83. State and prove the theorem of parallel axis.} \]

(4 marks)

**Ans.** **Theorem of parallel axis:** The moment of inertia of a body about an axis is equal to the sum of (i) its moment of inertia about a parallel axis through its centre of mass and (ii) the product of the mass of the body and the square of the distance between the two axes.

**Proof:** Let \( I_{CM} \) be the moment of inertia (MI) of a body of mass \( M \) about an axis through its centre of mass \( C \), and \( I \) be its MI about a parallel axis through any point \( O \). Let \( h \) be the distance between the two axes.

Consider an infinitesimal mass element \( dm \) of the body at a point \( P \). It is at a perpendicular distance \( CP \) from the rotation axis through \( C \) and a perpendicular distance \( OP \) from the parallel axis through \( O \). The MI of the element about the axis through \( C \) is \( CP^2 dm \). Therefore, the MI of the body about the axis through the CM is \( I_{CM} = \int CP^2 dm \). Similarly, the MI of the body about the parallel axis through \( O \) is \( I = \int OP^2 dm \).

\[ \text{Fig. 1.33 : Theorem of parallel axis} \]

Draw \( PQ \) perpendicular to \( OC \) produced, as shown in the figure. Then, from the figure,

\[ I = \int OP^2 dm \]
\[ = \int (OQ^2 + PQ^2) dm \]
\[ = \int [(OC + CQ)^2 + PQ^2] dm \]
\[ = \int (OC^2 + 2OC \cdot CQ + CQ^2 + PQ^2) dm \]
\[ = \int (OC^2 + 2OC \cdot CQ + CP^2) dm \]  
\[ \quad (\because \ CQ^2 + PQ^2 = CP^2) \]
\[ = \int OC^2 dm + \int 2OC \cdot CQ dm + \int CP^2 dm \]
\[ = OC^2 \int dm + 2OC \int CQ dm + \int CP^2 dm \]

Since, \( OC = h \) is constant and \( \int dm = M \) is the mass of the body,
\[ I = Mh^2 + 2h \int CQ dm + I_{CM} \]

Now, from the definition of centre of mass, the integral \( \int CQ dm \) gives mass \( M \) times a coordinate of the CM with respect to the origin \( C \). Since \( C \) is itself the CM, this coordinate is zero and so also the integral.

\[ \therefore \ I = I_{CM} + Mh^2 \]

This proves the theorem of parallel axis.
For the point P in Fig. 1.33 above, we had to extend OC to Q to meet the perpendicular PQ. What will happen to the expression for \( I \) if the point P lies on OC?

There will be no change in the expression for the MI (I) about the parallel axis through O.

**Q. 84. State and prove the theorem of perpendicular axes about moment of inertia.**

**Ans.** Theorem of perpendicular axes: The moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two mutually perpendicular axes in its plane and through the point of intersection of the perpendicular axis and the lamina.

**Proof:** Let O\(x\) and O\(y\) be two perpendicular axes in the plane of the lamina and O\(z\), an axis perpendicular to its plane. Consider an infinitesimal mass element \(dm\) of the lamina at the point \(P(x, y)\). MI of the lamina about the \(z\)-axis,

\[
I_z = \int OP^2 dm
\]

The element is at perpendicular distance \(y\) and \(x\) from the \(x\)- and \(y\)- axes respectively. Hence, the moments of inertia of the lamina about the \(x\)- and \(y\)-axes are, respectively,

\[
I_x = \int y^2 dm \quad \text{and} \quad I_y = \int x^2 dm
\]

Since \(OP^2 = y^2 + x^2\),

\[
I_z = \int OP^2 dm = \int (y^2 + x^2) dm
= \int y^2 dm + \int x^2 dm
\]

\[
\therefore \quad I_z = I_x + I_y
\]

This proves the theorem of perpendicular axes.

**Q. 85. State the conditions under which the theorems of parallel axis and perpendicular axes are applicable.**

**Ans.** The theorem of parallel axis is applicable to any body of arbitrary shape. The moment of inertia (MI) of the body about an axis through the centre of mass should be known, say, \(I_{CM}\). Then, the theorem can be used to find the MI, \(I\), of the body about an axis parallel to the above axis. If the distance between the two axes is \(h\),

\[
I = I_{CM} + Mh^2 \quad \ldots (1)
\]

The theorem of perpendicular axes is applicable to a plane lamina only. The moment of inertia \(I_z\) of a plane lamina about an axis–the \(z\) axis–perpendicular to its plane is equal to the sum of its moments of inertia \(I_x\) and \(I_y\) about two mutually perpendicular axes \(x\) and \(y\) in its plane and through the point of intersection of the perpendicular axis and the lamina.

\[
I_z = I_x + I_y \quad \ldots (2)
\]

**Q. 86. About which axis of rotation is the radius of gyration of a body the least?**

**Ans.** The radius of gyration of a body is the least about an axis through the centre of mass (CM) of the body.

From the parallel axis theorem, we know that a given body has the smallest possible moment of inertia about an axis through its CM. The radius of gyration of a body about a given axis is directly proportional to the square root of its moment of inertia about that axis. Hence, the conclusion.

\[
\{ \text{OR} \quad I = I_{CM} + Mh^2. \quad \therefore \quad Mk^2 = Mk_{CM}^2 + Mh^2. \\
\therefore \quad k^2 = k_{CM}^2 + h^2, \quad \text{which shows that} \ k \ \text{is minimum, equal to} \ k_{CM} \ \text{when} \ h = 0. \}
\]

**Q. 87. State an expression for the moment of inertia of a thin uniform rod about an axis through its centre and perpendicular to its length.**

Hence deduce the expression for its moment of inertia about an axis through its one end and perpendicular to its length.

**Ans.** The expression for the moment of inertia of a thin uniform rod about its transverse symmetry
Hence, deduce the expression for its moment of inertia about a parallel axis through one end. Also deduce the expressions for the corresponding radii of gyration.

 Ans. (3 marks)

(1) **MI about a transverse axis through centre** : Consider a thin uniform rod AB of mass $M$ and length $L$, rotating about a transverse axis through its centre C, Fig. 1.35. C is also its centre of mass (CM).

The moment of inertia (MI) of the rod about a transverse axis through C is

$$I_{CM} = \frac{ML^2}{12} \quad \ldots (1)$$

(2) **MI about a transverse axis through one end** : Let $I$ be its MI about a transverse axis through its end A. By the theorem of parallel axis,

$$I = I_{CM} + Mh^2 \quad \ldots (2)$$

In this case,

$$h = \text{distance between the parallel axes} = \frac{L}{2}$$

$$I = \frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2$$

$$= \frac{ML^2}{12} + \frac{ML^2}{4}$$

$$= \frac{ML^2}{12}(1 + 3) = \frac{ML^2}{3} \quad \ldots (3)$$

(3) **Radii of gyration** : The radius of gyration of the rod about its transverse symmetry axis is

$$k = \sqrt{\frac{I}{M}} = \sqrt{\frac{L^2}{3}} = \frac{L}{\sqrt{3}}$$

Q. 88. State the expression for the MI of a thin spherical shell (i.e., a thin-walled hollow sphere) about its diameter. Hence obtain the expression for its MI about a tangent. (1 mark)

Ans. Consider a uniform, thin-walled hollow sphere radius $R$ and mass $M$. An axis along its diameter is an axis of spherical symmetry through its centre of mass. The MI of the thin spherical shell about its diameter is

$$I_{CM} = \frac{2}{3}MR^2$$

Let $I$ be its MI about a tangent parallel to the diameter. Here, $h = R = \text{distance between the two axes}$. Then, according to the theorem of parallel axis,

$$I = I_{CM} + Mh^2$$

$$= \frac{2}{3}MR^2 + MR^2 = \frac{5}{3}MR^2$$

[Note : The corresponding radii of gyration are

$$k_{CM} = \sqrt{\frac{I_{CM}}{M}} = \sqrt{\frac{2}{3}} R \approx 0.8165 R \text{ and}$$

$$k = \sqrt{\frac{I}{M}} = \sqrt{\frac{5}{3}} R \approx 1.291 R$$]

Q. 89. Calculate the moment of inertia by direct integration of a thin uniform rod of mass $M$ and length $L$ about an axis perpendicular to the rod and passing through the rod at $L/3$, as shown below. (3 marks)

Check your answer with the parallel-axis theorem. (2 marks)

Ans. **Method of direct integration** : Consider a thin uniform rod of mass $M$ and length $L$. The axis of rotation is perpendicular to the rod and passing through the rod at $L/3$. We consider the origin of coordinates to be at this point and the $x$-axis to be along the rod,

$$k = \sqrt{\frac{I}{M}} = \sqrt{\frac{L^2}{12}} = \frac{L}{2\sqrt{3}} \quad \ldots (5)$$
Fig. 1.37. Since the mass density is constant, the linear mass density is 
\[ \lambda = \frac{M}{L} \]
An element of the rod has mass \( dm \) and length \( dl = dx \).

\[ \therefore \quad dm = \lambda dl = \lambda dx \]
If the distance of each mass element from the axis is given by the variable \( x \), the moment of inertia of an element about the axis of rotation is
\[ dI = x^2 dm \]
Since the rod extends from \( x = -L/3 \) to \( x = 2L/3 \), the MI of the rod about the axis is
\[ I = \int_{-L/3}^{2L/3} x^2 dm = \int_{-L/3}^{2L/3} x^2 (\lambda dx) \]
\[ = \lambda \int_{-L/3}^{2L/3} x^2 dx = \lambda \left[ \frac{x^3}{3} \right]_{-L/3}^{2L/3} \]
\[ = \frac{\lambda}{3} \left[ \left( \frac{2L}{3} \right)^3 - \left( -\frac{L}{3} \right)^3 \right] = \frac{L^3}{9} \times \frac{27}{27} = \frac{1}{2} ML^2 \]

Method of parallel-axis: The MI of the thin rod about a transverse axis through its CM is

\[ I_{CM} = \frac{1}{12} ML^2 \]
The given axis of rotation is at a distance
\[ h = \frac{L}{2} - \frac{L}{3} = \frac{L}{6} \]
from the transverse symmetry axis.
Therefore, the MI of the rod about the given axis is
\[ I = I_{CM} + Mh^2 = \frac{1}{12} ML^2 + M \left( \frac{L}{6} \right)^2 = \frac{1}{9} ML^2 \]
the same as arrived at by direct integration method.

Q. 90. State an expression for the moment of inertia of a thin ring about its transverse symmetry axis. Hence deduce the expression for its moment of inertia about a tangential axis perpendicular to its plane. Also deduce the expressions for the corresponding radius of gyration. (3 marks each)

Ans.

1) MI about the transverse symmetry axis: Consider a thin ring (or hoop) of radius \( R \) and mass \( M \). The axis of rotation through its centre C is perpendicular to its plane. C is also its centre of mass (CM). It is assumed that the radial thickness of the ring is so small as to be completely negligible in comparison to radius \( R \).

![Image of ring with axes](image)

The MI of the ring about the transverse symmetry axis is
\[ I_{CM} = MR^2 \]  
...(1)

Radius of gyration: The radius of gyration of the ring about the transverse symmetry axis is
\[ k = \sqrt{I_{CM}/M} = \sqrt{R^2} = R \]  
...(2)

2) MI about a tangent perpendicular to its plane: Let \( I \) be its MI about a parallel axis, tangent to the ring. Here, \( h = R \) = distance between the two axes.

By the theorem of parallel axis,
\[ I = I_{CM} + Mh^2 \]  
...(3)
\[ = MR^2 + MR^2 = 2MR^2 \]  
...(4)

Radius of gyration: The radius of gyration of the ring about a transverse tangent is
\[ k = \sqrt{I/M} = \sqrt{2R^2} = \sqrt{2R} \]  
...(5)

Q. 91. Assuming the expression for the moment of inertia of a ring about its transverse symmetry axis, obtain the expression for its moment of inertia about (1) a diameter (2) a tangential axis in its plane. Also deduce the expressions for the corresponding radii of gyration. (3 marks each)
Ans. Let \( M \) be the mass of a thin ring of radius \( R \). Let \( I_{\text{CM}} \) be the moment of inertia (MI) of the ring about its transverse symmetry axis. Then,
\[
I_{\text{CM}} = MR^2
\]
... (1)

(1) MI about a diameter: Let \( x \)- and \( y \)-axes be along two perpendicular diameters of the ring as shown in Fig. 1.39. Let \( I_x \), \( I_y \) and \( I_z \) be the moments of inertia of the ring about the \( x \), \( y \) and \( z \) axes, respectively.

Radius of gyration: The radius of gyration of the ring for rotation about a tangent in its plane is
\[
k = \sqrt{\frac{I}{M}} = \sqrt{\frac{3}{2} R^2} = \frac{3}{2} R
\]
... (8)

Q. 92. State an expression for the MI of a thin uniform disc about a transverse axis through its centre. Hence, derive an expression for the MI of the disc about its tangent perpendicular to the plane. Deduce the expressions for the corresponding radii of gyration. (3 marks)

Ans.
(1) MI about the transverse symmetry axis: Consider a thin uniform disc of radius \( R \) and mass \( M \). The axis of rotation through its centre \( C \) is perpendicular to its plane. \( C \) is also its centre of mass (CM).

The MI of the disc about the transverse symmetry axis is
\[
I_{\text{CM}} = \frac{1}{2} MR^2
\]
... (1)

Radius of gyration: The radius of gyration of the disc for rotation about a tangent in its plane is
\[
k = \sqrt{\frac{I}{M}} = \sqrt{\frac{3}{2} R^2} = \frac{3}{2} R
\]
... (2)

(2) MI about a tangent in its plane: Let \( I \) be its MI about an axis in plane of the ring, i.e., parallel to a diameter, and tangent to it. Here, \( h = R \) and
\[
I_{\text{CM}} = I_x = \frac{1}{2} MR^2.
\]

By the theorem of parallel axis,
\[
I = I_x + Mh^2 = \frac{1}{2} MR^2 + MR^2 = \frac{3}{2} MR^2
\]
... (7)
(2) **MI about a tangent perpendicular to its plane**: Let $I$ be the MI of the disc about a tangent perpendicular to its plane.

According to the theorem of parallel axis,
\[ I = I_{CM} + Mh^2 \]  
\[ = \frac{MR^2}{2} + MR^2 \quad (\because \ I_{CM} = \frac{MR^2}{2}, \ h = R) \]
\[ \therefore \ I = \frac{3}{2} MR^2 \]  
\[ \therefore \ I = \frac{3}{2} MR^2 \]  

**Radius of gyration**: The radius of gyration of the disc for the given rotation axis is
\[ k = \sqrt{\frac{I}{M}} = \sqrt{\frac{3R^2}{2}} = \sqrt{\frac{3}{2}R} \]  

Q. 93. **Assuming the expression for the moment of inertia of a thin uniform disc about a transverse axis through its centre, obtain an expression for its moment of inertia about any diameter. Hence, write the expression for the corresponding radius of gyration.** (3 marks)

**Ans.** Consider a thin uniform disc of mass $M$ and radius $R$ in the $xy$ plane, as shown in Fig. 1.42. Let $I_x$, $I_y$, and $I_z$ be the moments of inertia of the disc about the $x$, $y$, and $z$ axes respectively. But, $I_x = I_y$, since each represents the moment of inertia (MI) of the disc about its diameter and, by symmetry, the MI of the disc about any diameter is the same.

As $I_z$ is the MI of the disc about the $z$-axis through its centre and perpendicular to its plane,
\[ I_z = \frac{1}{2} MR^2 \]  

According to the theorem of perpendicular axes,
\[ I_x = I_y \]  
\[ \therefore \ I_x = 2I_y \]  
\[ \therefore \ I_x = \frac{1}{2} MR^2 = 2I_y \]  
\[ \therefore \ I_y = \frac{1}{4} MR^2 \]  

**Radius of gyration**: The radius of gyration of the disc for rotation about its diameter is
\[ k = \sqrt{\frac{I}{M}} = \sqrt{\frac{R^2}{4}} = \frac{R}{2} \]  

Q. 94. **Given the moment of inertia of a thin uniform disc about its diameter to be $\frac{1}{4} MR^2$, where $M$ and $R$ are respectively the mass and radius of the disc, find its moment of inertia about an axis normal to the disc and passing through a point on its edge.** (2 marks)

**Ans.** Consider a thin uniform disc of mass $M$ and radius $R$ in the $xy$ plane [see Fig. 1.42]. Let $I_x$, $I_y$, and $I_z$ be the moments of inertia of the disc about the $x$, $y$, and $z$ axes respectively.

Now, $I_x = I_y$ since each represents the moment of inertia (MI) of the disc about its diameter and, by symmetry, the MI of the disc about any diameter is the same.

\[ \therefore \ I_x = I_y = \frac{1}{4} MR^2 \]  
(Given)

According to the theorem of perpendicular axes,
\[ I_z = I_x + I_y = 2 \left( \frac{1}{4} MR^2 \right) = \frac{1}{2} MR^2 \]

Let $I$ be the MI of the disc about a tangent normal to the disc and passing through a point on its edge (i.e., a tangent perpendicular to its plane) [see Fig. 1.41]. According to the theorem of parallel axis,
\[ I = I_{CM} + Mh^2 \]

Here, $I_{CM} = \frac{1}{2} MR^2$ and $h = R$.

\[ \therefore \ I = \frac{1}{2} MR^2 + MR^2 = \frac{3}{2} MR^2 \]

which is the required expression.
Q. 95. Assuming the expression for the moment of inertia of a thin uniform disc about its diameter, show that the moment of inertia of the disc about a tangent in its plane is $\frac{5}{4}MR^2$. Write the expression for the corresponding radius of gyration.

**Ans.** Let $M$ be the mass and $R$ be the radius of a thin uniform disc. Let $I_{CM}$ be the moment of inertia (MI) of the disc about a diameter. Then,

$$I_{CM} = \frac{1}{4}MR^2 \quad \ldots \quad (1)$$

Let $I$ be its MI about a tangent parallel to the diameter. Here, $h = R = \text{distance between the two axes.}$ According to the theorem of parallel axis,

$$I = I_{CM} + Mh^2 \quad \ldots \quad (2)$$

$$\therefore I = I_{CM} + MR^2 = \frac{1}{4}MR^2 + MR^2$$

$$\therefore I = \frac{5}{4}MR^2 \quad \ldots \quad (3)$$

**Radius of gyration :** The radius of gyration $k$ of the disc for rotation about a tangent in its plane is

$$k = \sqrt{\frac{I}{M}} = \sqrt{\frac{5R^2}{4}} = \frac{\sqrt{5}}{2}R \quad \ldots \quad (4)$$

Q. 96. State the expressions for the moment of inertia of a solid cylinder of uniform cross section about (1) an axis through its centre and perpendicular to its length

(2) its own axis of symmetry. \hspace{1cm} (1 mark)

OR

State the expressions for the MI of a solid cylinder about (1) a transverse symmetry axis (2) its cylindrical symmetry axis. \hspace{1cm} (2 marks)

**Ans.** Consider a solid cylinder of uniform density, length $L$, radius $R$ and total mass $M$.

![Transverse symmetry axis of a cylinder and its cylindrical symmetry axis](image)

1. MI about transverse symmetry axis :

$$I_{CM} = M\left(\frac{R^2}{4} + \frac{L^2}{12}\right)$$

2. MI about cylindrical symmetry axis :

$$I_{CM} = \frac{1}{2}MR^2$$

3. Radii of gyration : The radius of gyration of the cylinder about its transverse symmetry axis is

$$k = \sqrt{\frac{I_{CM}}{M}} = \sqrt{\frac{R^2}{4} + \frac{L^2}{12}}$$

The radius of gyration of the cylinder about its cylindrical symmetry axis is

$$k = \sqrt{\frac{I_{CM}}{M}} = \frac{R}{\sqrt{2}} \quad \sqrt{\frac{R}{2}}$$

[Notes : (1) For $R \ll L$, a solid cylinder can be approximated as a thin rod, and the expression for the MI about its transverse symmetry axis reduces to the corresponding expression for a thin rod, viz., $ML^2/12$. (2) The MI of a solid cylinder about its cylindrical symmetry axis is the same as that of a disc about its transverse symmetry axis and having the same mass and radius.]

Q. 97. Assuming the expression for the moment of inertia of a uniform solid cylinder about a transverse symmetry axis, obtain the expression for its moment of inertia about a transverse axis through its one end. \hspace{1cm} (2 marks)

**Ans.** Let $M$ be the mass, $L$ the length and $R$ the radius of a uniform solid cylinder. Let $I_{CM}$ be the moment of inertia (MI) of the cylinder about a transverse symmetry axis. Then,
\[ I_{CM} = M \left( \frac{R^2}{4} + \frac{L^2}{12} \right) \quad \ldots (1) \]

Let \( I \) be its MI about a parallel, transverse axis through its one end. Here, \( h = \frac{L}{2} = \) distance between the two axes.

By the theorem of parallel axis,
\[ I = I_{CM} + Mh^2 \quad \ldots (2) \]
\[ \therefore I = M \left( \frac{R^2}{4} + \frac{L^2}{12} \right) + M \left( \frac{L}{2} \right)^2 \quad \ldots (3) \]
\[ = MR^2/4 + ML^2/12 + ML^2/4 \]
\[ = M \left( \frac{R^2}{4} + \frac{L^2}{3} \right) \quad \ldots (4) \]

Q. 98. State an expression for the moment of inertia of a solid sphere about its diameter. Write the expression for the corresponding radius of gyration. (2 marks)

Ans. Consider a solid sphere of uniform density, radius \( R \) and mass \( M \). An axis along its diameter is an axis of spherical symmetry through its centre of mass.

\[ I_{CM} = \frac{2}{5} MR^2 \]

The corresponding radius of gyration is
\[ k = \sqrt{\frac{I_{CM}}{M}} = \sqrt{\frac{2}{5} R^2} = \frac{\sqrt{2}}{\sqrt{5}} R \]

Q. 99. A uniform solid sphere of mass 15 kg has radius 0.1 m. What is its moment of inertia about a diameter? (1 mark)

Ans. Moment of inertia of the sphere about a diameter
\[ = \frac{2}{5} MR^2 = \frac{2}{5} \times 15 \times (0.1)^2 = 6 \times 10^{-2} \text{ kg\cdotm}^2 \]

Q. 100. Assuming the expression for the MI of a uniform solid sphere about its diameter, obtain the expression for its moment of inertia about a tangent. (1 mark)

Ans. Let \( M \) be the mass of a uniform solid sphere of radius \( R \). Let \( I_{CM} \) be its MI about any diameter.

Then, \( I_{CM} = \frac{2}{5} MR^2 \).

Q. 101. The moment of inertia of a uniform solid sphere about a diameter is 2 kg\cdotm^2. What is its moment of inertia about a tangent? (1 mark)
Q. 102. The radius of gyration of a uniform solid sphere of radius \( R \) is \( \sqrt{\frac{2}{5}} \) \( R \) for rotation about its diameter. Show that its radius of gyration for rotation about a tangential axis of rotation is \( \sqrt{\frac{7}{5}} R \).

Ans. Let the mass of the uniform solid sphere of radius \( R \) be \( M \). Let \( I_{CM} \) and \( k_d \) be its MI about any diameter and the corresponding radius of gyration, respectively. Then,

\[
I_{CM} = Mk_d^2 = \frac{2}{3} MR^2 \quad \text{(} \because k_d = \sqrt{\frac{2}{5}} R, \text{ given)}
\]

Let \( I \) and \( k_t \) be its MI about a parallel tangential axis and the corresponding radius of gyration, respectively. Here, \( h = R = \text{distance between the two axes.} \)

\[
\therefore \quad I = Mk_t^2
\]

By the theorem of parallel axis,

\[
I = I_{CM} + Mh^2
\]

\[
\therefore \quad Mk_t^2 = \frac{2}{3} MR^2 + MR^2 \quad \therefore \quad k_t^2 = \frac{2}{5} R^2 + R^2 = \frac{7}{5} R^2
\]

\[
\therefore \quad k_t = \sqrt{\frac{7}{5}} R
\]

Q. 103. State the expression for the MI of a thin spherical shell (i.e., a thin-walled hollow sphere) about its diameter. Hence obtain the expression for its MI about a tangent.

Ans. Consider a uniform, thin-walled hollow sphere radius \( R \) and mass \( M \). An axis along its diameter is an axis of spherical symmetry through its centre of mass. The MI of the thin spherical shell about its diameter is

\[
I_{CM} = \frac{2}{3} MR^2
\]

Let \( I \) be its MI about a tangent parallel to the diameter. Here, \( h = R = \text{distance between the two axes.} \) Then, according to the theorem of parallel axis,

\[
I = I_{CM} + Mh^2
\]

\[
= \frac{2}{3} MR^2 + MR^2 = \frac{5}{3} MR^2
\]

[Note: The corresponding radii of gyration are

\[
k_{CM} = \sqrt{\frac{I_{CM}}{M}} = \sqrt{\frac{2}{3} R} \approx 0.8165 \text{ } \text{ } \text{ } R \text{ and}
\]

\[
k = \sqrt{\frac{I}{M}} = \sqrt{\frac{5}{3} R} \approx 1.291 \text{ } \text{ } R
\]

Q. 104. Find the ratio of the radius of gyration of a solid sphere about its diameter to the radius of gyration of a hollow sphere about its tangent, given that both the spheres have the same radius.

(2 marks)

Ans. The radius of gyration of a body about a given axis, \( k = \sqrt{\frac{I}{M}} \), where \( M \) and \( I \) are respectively the mass of the body and its moment of inertia (MI) about the axis.

For a solid sphere rotating about its diameter,

\[
k_{SS} = \sqrt{\frac{2}{3} R}
\]

For a hollow sphere rotating about its diameter,

\[
i_{HS} = \frac{2}{3} MR^2
\]

For a hollow sphere rotating about its tangent,

\[
i'_{HS} = \frac{2}{3} MR^2 + MR^2 = \frac{5}{3} MR^2
\]

so that, its radius of gyration for rotation about a tangent is

\[
k'_{HS} = \sqrt{\frac{5}{3}} R
\]

The required ratio, \( \frac{k_{SS}}{k_{HS}} = \sqrt{\frac{2}{3} \times \frac{3}{5}} = \frac{\sqrt{6}}{5} \)

Q. 105. Calculate the moment of inertia by direct integration of a thin uniform rectangular plate of mass \( M \), length \( l \) and breadth \( b \) about an axis passing through its centre and parallel to its breadth.

(3 marks)

Ans. Consider a thin uniform rectangular plate of mass \( M \), length \( l \) and breadth \( b \). The axis of rotation passes through its centre and is parallel to its breadth. We consider the origin of coordinates to be at the centre of the plate and orient the axes as shown in Fig. 1.48. Since the plate is thin, we can
take the mass as distributed entirely in the $xy$-plane. Then, the surface mass density is constant and equal to 

$$\sigma = \frac{M}{A} = \frac{M}{lb}$$

A rectangular element of the plate, shown shaded, has mass $dm$, length $b$ and breadth $dy$.

$$\therefore dm = \sigma dA = \sigma (b \, dy)$$

If the distance of each element from the rotation axis is given by the variable $y$, the moment of inertia of an element about the axis of rotation is

$$dI_x = y^2 dm$$

Since the rod extends from $y = -l/2$ to $y = l/2$, the MI of the thin plate about the axis is

$$I_x = \int_{-l/2}^{l/2} dI_x = \int_{-l/2}^{l/2} y^2 (\sigma b dy)$$

$$= \sigma b \int_{-l/2}^{l/2} y^2 dy = \sigma b \left[ \frac{y^3}{2} \right]_{-l/2}^{l/2}$$

$$= \frac{\sigma b}{3} \left[ \left( \frac{l}{2} \right)^3 - \left( -\frac{l}{2} \right)^3 \right] = \frac{Mb^2}{3l} \times \frac{l^3}{4} = \frac{1}{12} Mb^2$$

**Notes:**

1. The MI of a thin rectangular plate about an axis passing through its centre and parallel to its length (i.e., about the $y$-axis in Fig. 1.48) is $I_y = \frac{1}{12} Mb^2$.

Then, by the theorem of perpendicular axes, the MI of a thin plate about its transverse symmetry axis (i.e., about the $z$-axis in Fig. 1.48) is

$$I_z = I_x + I_y = \frac{1}{12} M (l^2 + b^2)$$

2. Suppose, for a rectangular bar of sides $l$, $b$ and $w$, we take the origin of coordinates at the centre of mass of the bar and the $x$, $y$ and $z$ axes parallel to the respective sides. Then, $I_z = \frac{1}{12} M (b^2 + w^2)$, $I_y = \frac{1}{12} M (w^2 + l^2)$ and

$$I_z = \frac{1}{12} M (l^2 + b^2)$$

You will need to recall this in Chapter 5.

**Q. 106.** State the MI of a thin rectangular plate—of mass $M$, length $l$ and breadth $b$—about an axis passing through its centre and parallel to its length. Hence find its MI about a parallel axis along one edge.  

(2 marks)

**Ans.** Consider a thin rectangular plate of mass $M$, length $l$ and breadth $b$. The MI of the plate about an axis passing through its centre and parallel to its edge of length $l$ is

$$I_{CM, \text{length}} = \frac{1}{2} Mb^2$$

For a parallel axis along its one edge, $h = \frac{1}{2} b$. Therefore, by the theorem of parallel axis, the MI about this axis is

$$I_{\text{edge, length}} = I_{CM, \text{length}} + Mh^2$$

$$= \frac{1}{12} Mb^2 + M \left( \frac{b}{2} \right)^2 = \frac{1}{3} Mb^2$$

**Q. 107.** State the MI of a thin rectangular plate—of mass $M$, length $l$ and breadth $b$—about its transverse axis passing through its centre. Hence find its MI about a parallel axis through the midpoint of edge of length $b$.  

(2 marks)

**Ans.** Consider a thin rectangular plate of mass $M$, length $l$ and breadth $b$. The MI of the plate about its transverse axis passing through its centre is

$$I_{CM} = \frac{1}{12} M (l^2 + b^2)$$

For a parallel axis through the midpoint of its breadth, $h = \frac{1}{2} l$. Therefore, by the theorem of parallel axis, the MI about this axis is
\[ I = I_{CM} + Mh^2 = \frac{1}{12} M(l^2 + b^2) + M \left( \frac{l}{2} \right)^2 \]

\[ = M \left( \frac{1}{3} l^2 + \frac{1}{12} b^2 \right) \]

Fig. 1.50 : Transverse axis passing through its centre of a thin rectangular plate and a parallel axis

Q. 108. A uniform solid right circular cone of base radius \( R \) has mass \( M \). Prove that the moment of inertia of the cone about its central symmetry axis is \( \frac{3}{10} MR^2 \). (3 marks)

Ans. Consider a uniform solid right circular cone of mass \( M \), base radius \( R \) and height \( h \). The axis of rotation passes through its centre and the vertex, Fig. 1.51. Its constant mass density is

\[ \rho = \frac{M}{V} = \frac{M}{\frac{1}{3} \pi R^2 h} \]

\[ dm = \rho dV = \rho (\pi r^2 dz) = \frac{3M}{\pi R^2 h} (\pi r^2 dz) = \frac{3M}{R^2 h} r^2 dz \]

and \( \frac{r}{R} = \frac{z}{h} \)

\:: \: r = \frac{z}{h} \) and \( dm = \frac{3M z^2}{h} \frac{h}{h^2} dz = \frac{3M}{h} z^2 dz \)

The moment of inertia of an elemental disc about the axis of rotation is

\[ dl = \frac{1}{2} r^2 dm = \frac{1}{2} \left( \frac{R^2 z^2}{h^2} \right) 3M \frac{h}{h^2} z^2 dz = \frac{3M R^2}{2} \frac{h^5}{h^5} z^4 dz \]

Since the cone extends from \( z = 0 \) to \( z = h \), the MI of the cone about its central symmetry axis is

\[ I = \int dl = \frac{3M R^2}{2} \frac{h^5}{h^5} \int_0^h z^4 dz = \frac{3M R^2}{2} \frac{h^5}{h^5} \left[ \frac{z^5}{5} \right]_0^h \]

\[ = \frac{3M R^2 h^5}{2} \frac{h}{5} = \frac{3}{10} MR^2 \]

as required.

\[ \star Q. \ 109. \ A \ uniform \ disc \ and \ a \ hollow \ right \ circular \ cone \ have \ the \ same \ formula \ for \ moment \ of \ inertia \ for \ rotation \ about \ their \ corresponding \ central \ symmetry \ axes. \ Why \ is \ it \ so? \quad (1 \ mark) \]

Ans. The moment of inertia of a hollow right circular cone about its central symmetry axis is \( \frac{1}{2} MR^2 \), the same as that of a disc about its transverse symmetry axis. This is because the distribution of mass of the hollow cone about its central symmetry axis is the same as that of a disc.

\[ \star \]

Solved Problems 1.5.1 – 1.7

Q. 110. Solve the following:

(1) Calculate the moment of inertia of a ring of mass 500 g and radius 0.5 m about an axis of rotation passing through (i) its diameter (ii) a tangent perpendicular to its plane. (3 marks)

Solution:

Data : \( M = 500 \) g = 0.5 kg, \( R = 0.5 \) m

(i) The moment of inertia of the ring about its diameter

\[ = \frac{MR^2}{2} = \frac{0.5 \times (0.5)^2}{2} = 0.0625 \text{ kg m}^2 = 6.25 \times 10^{-2} \text{ kg m}^2 \]

(ii) The moment of inertia of the ring about a tangent perpendicular to its plane

\[ = 2MR^2 = 2 \times 0.5 \times (0.5)^2 = 0.25 \text{ kg m}^2 \]
(2) A metal ring of mass 1 kg has moment of inertia 10 kg·m² for rotation about its diameter. It is melted and recast into a thin uniform disc of the same radius. What will be the disc’s moment of inertia when rotated about its own axis? (2 marks)

Solution:

The MI of the thin ring about its diameter,

$$I_{\text{ring}} = \frac{1}{2} MR^2 = 1 \text{ kg·m}^2$$

Since the ring is melted and recast into a thin disc of the same radius R, the mass of the disc equals the mass of the ring, M.

The MI of the thin disc about its own axis (i.e., transverse symmetry axis) is

$$I_{\text{disc}} = \frac{1}{2} MR^2 = I_{\text{ring}}$$

∴ $$I_{\text{disc}} = 1 \text{ kg·m}^2$$

(3) A thin uniform rod 1 m long has mass 1 kg. Find its moment of inertia and radius of gyration for rotation about a transverse axis through a point midway between its centre and one end. (3 marks)

Solution:

Data: $$M = 1 \text{ kg}, L = 1 \text{ m}$$

Let $$I_{\text{CM}}$$ and I be the moments of inertia of the rod about a transverse axis through its centre, and a parallel axis midway between its centre and one end.

Then, $$I_{\text{CM}} = \frac{ML^2}{12}$$ and $$h = \frac{L}{4}$$

By the theorem of parallel axis,

$$I = I_{\text{CM}} + Mh^2$$

$$= \frac{ML^2}{12} + \frac{ML^2}{16} = \frac{4ML^2 + 3ML^2}{48}$$

$$= \frac{7}{48} ML^2 = \frac{7}{48} (1)(1)^2 = 0.1458 \text{ kg·m}^2$$

If k is the radius of gyration about the parallel axis,

$$I = Mk^2$$ ∴ $$Mk^2 = \frac{7}{48} ML^2$$

∴ $$k = \sqrt{\frac{7}{48} L} = \frac{\sqrt{7}}{\sqrt{48}} \times 1 = 0.3818 \text{ m}$$

(4) The moment of inertia of a disc about an axis through its centre and perpendicular to its plane is 10 kg·m². Find its MI about a diameter. (2 marks)

Solution:

Data: $$I_z = 10 \text{ kg·m}^2$$

If the disc lies in the xy plane with its centre at the origin then, according to the theorem of perpendicular axes,

$$I_x + I_y = I_z$$

Since, $$I_x = I_y$$, $$2I_x = I_z$$

∴ Its MI about a diameter,

$$I_x = \frac{I_z}{2} = \frac{10}{2} = 5 \text{ kg·m}^2$$

(5) A solid cylinder of uniform density and radius 2 cm has a mass of 50 g. If its length is 12 cm, calculate its moment of inertia about an axis passing through its centre and perpendicular to its length. (2 marks)

Solution:

Data: $$M = 50 \text{ g}, R = 2 \text{ cm}, L = 12 \text{ cm}$$

$$I_{\text{CM}} = \frac{MR^2}{4} + \frac{ML^2}{12} = M \left( \frac{R^2}{4} + \frac{L^2}{12} \right)$$

$$= 50 \left[ \frac{(2)^2}{4} + \frac{(12)^2}{12} \right]$$

$$= 50 \left( \frac{4}{4} + \frac{144}{12} \right) = 50 (1 + 12)$$

$$= 50 \times 13 = 650 \text{ g·cm}^2$$

(6) A dumbbell is prepared by using a uniform rod of mass 60 g and length 20 cm. Two identical solid spheres of mass 50 g and radius 10 cm each are at the two ends of the rod. Calculate moment of inertia of the dumbbell for rotation about an axis passing through its centre and perpendicular to the length. (3 marks)

Solution:

Data: $$M_{\text{sph}} = 50 \text{ g}, R_{\text{sph}} = 10 \text{ cm}, M_{\text{rod}} = 60 \text{ g}, L_{\text{rod}} = 20 \text{ cm}$$

The MI of a solid sphere about its diameter is

$$I_{\text{sph, CM}} = \frac{2}{5} M_{\text{sph}} R_{\text{sph}}^2$$

The distance of the rotation axis (transverse symmetry axis of the dumbbell) from the centre of sphere, $$h = 30 \text{ cm}$$.

The MI of a solid sphere about the rotation axis,

$$I_{\text{sph}} = I_{\text{sph, CM}} + M_{\text{sph}} h^2$$
For the rod, the rotation axis is its transverse symmetry axis through CM.
The MI of a rod about this axis,
\[ I_{\text{rod}} = \frac{1}{12} M_{\text{rod}} L_{\text{rod}}^2 \]
Since there are two solid spheres, the MI of the dumbbell about the rotation axis is
\[ I = 2I_{\text{sph}} + I_{\text{rod}} \]
\[ = 2M_{\text{sph}} \left( \frac{2}{5} R_{\text{sph}}^2 + h^2 \right) + \frac{1}{12} M_{\text{rod}} L_{\text{rod}}^2 \]
\[ = 2(50) \left( \frac{2}{5} (10)^2 + (30)^2 \right) + \frac{1}{12} (60)(20)^2 \]
\[ = 100(40 + 900) + 5(400) = 94000 + 2000 \]
\[ = 96000 \text{ g·cm}^2 \]

(7) A compound object is formed of a thin rod and a disc attached at the end of the rod. The rod is 0.5 m long and has mass 2 kg. The disc has mass of 1 kg and its radius is 20 cm. Find the moment of inertia of the compound object about an axis passing through the free end of the rod and perpendicular to its length. (3 marks)

Solution :

Data : \( L = 0.5 \text{ m}, R = 0.2 \text{ m}, M_{\text{rod}} = 2 \text{ kg}, M_{\text{disk}} = 1 \text{ kg} \)

About a transverse axis through CM,
\[ I_{\text{CM,rod}} = \frac{1}{2} M_{\text{rod}} L_{\text{rod}}^2 \quad \text{and} \quad I_{\text{CM,disk}} = \frac{1}{2} M_{\text{disk}} R_{\text{disk}}^2 \]

The MI of the compound object about the given axis,
\[ I_{\text{total}} = I_{\text{rod}} + I_{\text{disk}} \]
\[ = \left[ I_{\text{CM,rod}} + M_{\text{rod}} \left( \frac{L}{2} \right)^2 \right] + \left[ I_{\text{CM,disk}} + M_{\text{disk}} (L + R)^2 \right] \]
\[ = \left( \frac{1}{12} M_{\text{rod}} L_{\text{rod}}^2 + \frac{1}{4} M_{\text{rod}} L_{\text{rod}}^2 \right) + \left[ \frac{1}{2} M_{\text{disk}} R_{\text{disk}}^2 + M_{\text{disk}} (L + R)^2 \right] \]
\[ = \frac{1}{3} M_{\text{rod}} L_{\text{rod}}^2 + M_{\text{disk}} \left[ \frac{1}{2} R_{\text{disk}}^2 + M_{\text{disk}} (L + R)^2 \right] \]
\[ = \frac{1}{3} (2)(0.5)^2 + (1) \left[ \frac{1}{2} (0.2)^2 + (0.5 + 0.2)^2 \right] \]
\[ = \frac{1}{6} \times 0.04 + 0.49 = 0.167 + 0.02 + 0.49 \]
\[ = 0.677 \text{ kg·m}^2 \]

(8) The radius of gyration of a body about an axis at 6 cm from its centre of mass is 10 cm. Find its radius of gyration about a parallel axis through its centre of mass. (2 marks)

Solution :

Let O be a point at 6 cm from the centre of mass of the body.

Let \( I = I_{\text{CM}} + Mh^2 \)

Also, \( I = Mk_1^2 \) and \( I_{\text{CM}} = Mk_{\text{CM}}^2 \)

\[ \therefore \ k^2 = k_{\text{CM}}^2 + h^2 \]

\[ \therefore \ 100 = k_{\text{CM}}^2 + 36 \quad \therefore \ k_{\text{CM}} = 8 \text{ cm} \]

The radius of gyration about a parallel axis through its centre of mass is 8 cm.

(9) The radius of gyration of a disc about its transverse symmetry axis is 2 cm. Determine its radius of gyration about a diameter. (2 marks)

Solution :

Data : \( k_{\text{CM}} = 2 \text{ cm} \)

Let \( M \) and \( R \) be the mass and radius of the disc. Let \( I_{\text{CM}} \) and \( k_{\text{CM}} \) be the MI and radius of gyration of the disc about its transverse symmetry axis. Let \( I \) and \( k \) be the MI and radius of gyration of the disc about its diameter. Then

\[ I_{\text{CM}} = \frac{MR^2}{2} = Mk_{\text{CM}}^2 \quad \text{and} \quad I = \frac{MR^2}{4} = Mk^2 \]

\[ \therefore \ k_{\text{CM}}^2 = \frac{R^2}{2} \quad \text{and} \quad k^2 = \frac{R^2}{4} \]

\[ \therefore \ \frac{k_{\text{CM}}^2}{4} \times \frac{2}{R^2} = \frac{1}{2} \]

\[ \therefore \ k^2 = \frac{k_{\text{CM}}^2}{2} \quad \therefore \ k = \frac{k_{\text{CM}}}{\sqrt{2}} \]

\[ \therefore \ k = \frac{2}{\sqrt{2}} = \sqrt{2} = 1.414 \text{ cm} \]
(10) Calculate the MI and rotational kinetic energy of a thin uniform rod of mass 10 g and length 60 cm when it rotates about a transverse axis through its centre at 90 rpm. \( \text{Solution :} \)

**Data :** \( M = 10 \text{ g} = 10^{-2} \text{ kg}, L = 60 \text{ cm} = 0.6 \text{ m}, \)

\( f = 90 \text{ rpm} = 90/60 \text{ Hz} = 1.5 \text{ Hz} \)

The MI of the rod about a transverse axis through its centre is

\[ I = \frac{ML^2}{12} = \frac{(10^{-2})(0.6)^2}{12} = 3 \times 10^{-4} \text{ kg} \cdot \text{m}^2 \]

Angular speed, \( \omega = 2\pi f = 2 \times 3.142 \times 1.5 \)

= 9.426 \text{ rad/s} \]

Rotational KE = \( \frac{1}{2}Io^2 = \frac{1}{2}(3 \times 10^{-4})(9.426)^2 \)

= 0.01333 J

(11) A thin rod of uniform cross section is made up of two sections made of wood and steel. The wooden section has length 50 cm and mass 0.6 kg. The steel section has length 30 cm and mass 3 kg. Find the moment of inertia of the rod about a transverse axis passing through the junction of the two sections. \( \text{Solution :} \)

**Data :** \( L_1 = 0.5 \text{ m}, M_1 = 0.6 \text{ kg}, L_2 = 0.3 \text{ m}, M_1 = 3 \text{ kg} \)

The moment of inertia of a thin rod about a transverse axis through its end is \( \frac{ML^2}{3} \).

Therefore, the MI of the composite rod about a transverse axis through the junction of the two sections,

\[ I = \frac{M_1L_1^2}{3} + \frac{M_2L_2^2}{3} \]

\[ = \frac{0.6(0.5)^2}{3} + \frac{3(0.3)^2}{3} \]

\[ = 0.05 + 0.09 = 0.14 \text{ kg} \cdot \text{m}^2 \]

(12) The mass and the radius of the Moon are, respectively, about \( \frac{1}{81} \) time and about \( \frac{1}{37} \) time those of the Earth. Given that the rotational period of the Moon is 27.3 days, compare the rotational kinetic energy of the Earth with that of the Moon. \( \text{Solution :} \)

**Data :** \( M_M = \frac{1}{81}M_E, R_M = \frac{1}{37}R_E, T_M = 27.3 \text{ days}, \)

\( T_E = 1 \text{ day} \)

Let \( I_E \) and \( I_M \) be the moments of inertia of the Earth and the Moon about their respective axes of rotation, and \( \omega_E \) and \( \omega_M \) be their respective rotational angular speeds. Assuming the Earth and the Moon to be solid spheres of uniform densities,

\[ I_E = \frac{2}{5}M_ER_E^2 \quad \text{and} \quad I_M = \frac{2}{5}M_MR_M^2 \]

Kinetic energy of rotation, \( E_{rot} = \frac{1}{2}Io^2 \)

\[ = \frac{1}{2}I \left( \frac{2\pi}{T} \right)^2 = \frac{2\pi^2}{T^2} \]

Therefore, the ratio of the rotational KE of the Earth to that of the Moon is

\[ \frac{E_{rot \text{ (Earth)}}}{E_{rot \text{ (Moon)}}} = \frac{I_E}{I_M} \left( \frac{T_M}{T_E} \right)^2 = \frac{M_E}{M_M} \left( \frac{R_E}{R_M} \right)^2 \left( \frac{T_M}{T_E} \right)^2 \]

\[ = 81 \times (3.7)^2 \times (27.3)^2 = 8.264 \times 10^5 \]

(13) A solid sphere of radius \( R \), rotating with an angular velocity \( \omega \) about its diameter, suddenly stops rotating and 75% of its KE is converted into heat. If \( c \) is the specific heat capacity of the material in SI units, show that the temperature of the sphere rises by \( \frac{3R^2\omega^2}{20c} \). \( \text{Solution :} \)

The MI of a solid sphere about its diameters,

\[ I = \frac{2}{5}MR^2 \]

where \( M \) is its mass.

The rotational KE of the sphere,

\[ E = \frac{1}{2}Io^2 = \frac{1}{2} \left( \frac{2}{5}MR^2 \right) \omega^2 \]

\[ = \frac{1}{5}MR^2\omega^2 \]

If \( \Delta \theta \) is the rise in temperature,
Q. 111. Define the angular momentum of a particle. (1 mark)

Ans. Definition: The angular momentum of a particle is defined as the moment of the linear momentum of the particle. If a particle of mass \( m \) has linear momentum \( \overrightarrow{p} \) (\( =mv \)), then the angular momentum of this particle with respect to a point \( O \) is a vector quantity defined as \( \overrightarrow{L} = \overrightarrow{r} \times \overrightarrow{p} = m(\overrightarrow{r} \times \overrightarrow{v}) \), where \( \overrightarrow{r} \) is the position vector of the particle with respect to \( O \). It is the angular analogue of linear momentum.

![Fig. 1.53: Angular momentum of a particle about O](image)

[Note: As the particle moves relative to \( O \) in the direction of its momentum \( \overrightarrow{p} \) (\( =mv \)), position vector \( \overrightarrow{r} \) rotates around \( O \). However, to have angular momentum about \( O \), the particle does not itself have to rotate around \( O \).]

Q. 112. State the dimensions and SI unit of angular momentum. (1 mark)

Ans.

1. **Dimensions**: [Angular momentum] = \([M^1L^2T^{-1}]\)
2. **SI unit**: The kilogram-metre^2/second (kg·m^2/s).

Q. 113. Obtain an expression for the angular momentum of a rigid body rotating with a constant angular velocity. (3 marks)

OR

Derive an expression that relates the angular momentum with the angular velocity of a rotating rigid body.

\[
Mc\Delta \theta = \frac{3}{4}E = \frac{3}{5}(\frac{1}{5}MR^2\omega^2)
\]

\[\therefore \Delta \theta = \frac{3R^2\omega^2}{20c}\]

Unit

1.8 Angular momentum

1.8.1 Expression for angular momentum (in terms of MI)

\[
\text{Ans. Consider a rigid body rotating with a constant angular velocity } \overrightarrow{\omega} \text{ about an axis through the point } O \text{ and perpendicular to the plane of the figure. All the particles of the body perform uniform circular motion about the axis of rotation with the same angular velocity } \overrightarrow{\omega}. \text{ Suppose that the body consists of } N \text{ particles of masses } m_1, m_2, \ldots, m_N, \text{ situated at perpendicular distances } r_1, r_2, \ldots, r_N \text{ respectively from the axis of rotation.}
\]

\[L_1 = r_1 \times p_1\]

\[\therefore L_1 = r_1 p_1 \sin \theta\]

where \( \theta \) is the smaller of the two angles between \( \overrightarrow{r}_1 \) and \( \overrightarrow{p}_1 \).

In this case, \( \theta = 90^\circ \) \( \therefore \) \( \sin \theta = 1 \)

\[L_1 = r_1 p_1 = r_1 m_1 r_1 \omega = m_1 r_1^2 \omega\]

Similarly, \( L_2 = m_2 r_2^2 \omega \), \( L_3 = m_3 r_3^2 \omega \), etc.

The angular momentum of the body about the given axis is

\[L = L_1 + L_2 + \ldots + L_N\]

\[= m_1 r_1^2 \omega + m_2 r_2^2 \omega + \ldots + m_N r_N^2 \omega\]

\[= (m_1 r_1^2 + m_2 r_2^2 + \ldots + m_N r_N^2) \omega\]

\[= \left( \sum_{i=1}^{N} m_i r_i^2 \right) \omega\]

\[\therefore L = I \omega\]
where \( I = \sum_{i=1}^{N} m_i r_i^2 \) = moment of inertia of the body about the given axis.

In vector form, \( \vec{L} = I \vec{\omega} \)

Thus, angular momentum

\[ = \text{moment of inertia} \times \text{angular velocity}. \]

[Note: Angular momentum is a vector quantity. It has the same direction as \( \vec{\omega} \).]

Q. 114. Express the kinetic energy of a rotating body in terms of its angular momentum. (1 mark)

Ans. The kinetic energy of a body of moment of inertia \( I \) and rotating with a constant angular velocity \( \omega \) is

\[ E = \frac{1}{2} I \omega^2 \]

The angular momentum of the body, \( L = I \omega \).

\[ . \quad E = \frac{1}{2} (I \omega) \omega = \frac{1}{2} I \omega^2 \]

This is the required relation.

Q. 115. Why do grinding wheels have large mass and moderate diameter? (1 mark)

Ans. A grinding wheel, used for abrasive machining operations (e.g., sharpening), is typically in the form of a heavy disc of moderate diameter. A grinding machine needs to have a high frequency of revolution but the machining operations exert braking torques on its wheel.

Angular momentum is directly proportional to mass. Hence, heavier the wheel, the greater is its angular momentum and lesser is the decelerating effect of the braking torques. Also, angular acceleration is inversely proportional to the moment of inertia. Since the wheel is made heavy, its diameter is kept moderate so that a large angular acceleration and high angular velocity can be achieved with a motor of given power.

### Solved Problems 1.8 – 1.8.1

Q. 116. Solve the following:

(1) The angular momentum of a body changes by 80 kg·m²/s when its angular velocity changes from 20 rad/s to 40 rad/s. Find the change in its kinetic energy of rotation. (2 marks)

Solution:

Data: \( \omega_1 = 20 \text{ rad/s}, \omega_2 = 40 \text{ rad/s} \)

If \( I \) is the MI of the body, its initial angular momentum is \( I \omega_1 \) and final angular momentum is \( I \omega_2 \).

Change in angular momentum

\[ = I \omega_2 - I \omega_1 = I(\omega_2 - \omega_1) \]

\[ . \quad 80 = I(40 - 20) \]

\[ . \quad I = 4 \text{ kg·m}^2 \]

Initial KE of the body is \( \frac{1}{2} I \omega_1^2 \) and final KE is \( \frac{1}{2} I \omega_2^2 \).

Change in KE

\[ = \frac{1}{2} I \omega_2^2 - \frac{1}{2} I \omega_1^2 = \frac{1}{2} I (\omega_2^2 - \omega_1^2) \]

\[ = \frac{1}{2} \times 4 \times (1600 - 400) = 2400 \text{ J} \]

(2) A wheel of moment of inertia 1 kg·m² is rotating at a speed of 40 rad/s. Due to the friction on the axis, the wheel comes to rest in 10 minutes. Calculate the angular momentum of the wheel, two minutes before it comes to rest. (2 marks)

Solution:

Data: \( I = 1 \text{ kg·m}^2, \omega_1 = 40 \text{ rad/s, } \omega_2 = 0 \text{ at } t = 10 \text{ minutes} = 60 \times 10 \text{ s} = 600 \text{ s}, \)

\( t' = 8 \text{ minutes} = 60 \times 8 \text{ s} = 480 \text{ s} \)

\( \omega_2 = \omega_1 + \alpha t \)

\[ . \quad \alpha = \frac{\omega_2 - \omega_1}{t} \]

\[ = \frac{0 - 40}{600} = - \frac{1}{15} \text{ rad/s}^2 \]

At time \( t' \),

\[ \omega_3 = \omega_1 + \alpha t' \]

\[ = 40 - \frac{1}{15} \times 480 = 40 - 32 = 8 \text{ rad/s} \]

\[ . \quad L = I \omega_3 = 1 \times 8 = 8 \text{ kg·m}^2\text{s} \]

This is the required angular momentum of the wheel.

(3) A flywheel rotating about an axis through its centre and perpendicular to its plane loses 100 J of energy on slowing down from 60 rpm to 30 rpm. Find its moment of inertia about the given axis and the change in its angular momentum. (3 marks)

Solution:

Data: \( f_1 = 60 \text{ rpm} = 60/60 \text{ rot/s} = 1 \text{ rot/s, } \)

\( f_2 = 30 \text{ rpm} = 30/60 \text{ rot/s} = 1/2 \text{ rot/s, } \Delta E = -100 \text{ J} \)
Q. 117. Obtain an expression for the torque on a body rotating with constant angular acceleration. OR

* Obtain an expression relating the torque with uniform angular acceleration for a rotating rigid body. (3 marks)

Ans. A torque acting on a body produces angular acceleration.

Consider a rigid body rotating about an axis passing through the point O and perpendicular to the plane of the figure. Suppose that a torque \( \vec{\tau} \) on the body produces uniform angular acceleration \( \vec{\alpha} \) along the axis of rotation.

The body can be considered as made up of \( N \) particles with masses \( m_1, m_2, \ldots, m_N \) situated at perpendicular distances \( r_1, r_2, \ldots, r_N \) respectively from the axis of rotation. \( \vec{\alpha} \) is the same for all the particles as the body is rigid. Let \( \vec{F}_1, \vec{F}_2, \ldots, \vec{F}_N \) be the external forces on the particles.

The torque \( \vec{\tau}_1 \), on the particle of mass \( m_1 \), is

\[ \vec{\tau}_1 = \vec{r}_1 \times \vec{F}_1 \]

\( \therefore \ \tau_1 = r_1 F_1 \sin \theta \)

where \( \theta \) is the smaller of the two angles between \( \vec{r}_1 \) and \( \vec{F}_1 \).

\( \therefore \ \tau_1 = r_1 F_1 \) since, in this case, \( \theta = 90^\circ \)

Now, \( F_1 = m_1 a_1 = m_1 r_1 \alpha \) ( \( \therefore \ a_1 = r_1 \alpha \) )

\( \therefore \ \tau_1 = r_1 (m_1 r_1 \alpha) = m_1 r_1^2 \alpha \)

Similarly, \( \tau_2 = m_2 r_2^2 \alpha, \ldots, \tau_N = m_N r_N^2 \alpha \)

The total torque on the body is

\[ \tau = \tau_1 + \tau_2 + \ldots + \tau_N = m_1 r_1^2 \alpha + m_2 r_2^2 \alpha + \ldots + m_N r_N^2 \alpha \]

\[ = \left( \sum_{i=1}^{N} m_i r_i^2 \right) \alpha = I \alpha \]

where \( I = \sum_{i=1}^{N} m_i r_i^2 \) is the moment of inertia of the body about the axis of rotation.

In vector form, \( \vec{\tau} = I \vec{\alpha} \)

This gives the required relation.

Angular acceleration \( \vec{\alpha} \) has the same direction as the torque \( \vec{\tau} \) and both of them are axial vectors along the rotation axis.

Q. 118. A torque of 4 N·m acting on a body of mass 1 kg produces an angular acceleration of 2 rad/s². What is the moment of inertia of the body? (1 mark)

Ans. The moment of inertia of the body.

\[ I = \frac{\tau}{\alpha} = \frac{4}{2} = 2 \text{ kg·m}^2 \]

Q. 119. Two identical rings are to be rotated about different axes of rotation as shown by applying torques so as to produce the same angular acceleration in both. How is it possible? (2 marks)
Q. 120. Two wheels have the same mass. First wheel is in the form of a solid disc of radius $R$ while the second is a disc with inner radius $r$ and outer radius $R$. Both are rotating with same angular velocity $\omega_0$ about transverse axes through their centres. If the first wheel comes to rest in time $t_1$ while the second comes to rest in time $t_2$, are $t_1$ and $t_2$ different? Why? (3 marks)

**Ans.** The moments of inertia of the two wheels about transverse axes through their centres are

$$I_1 = \frac{1}{2} MR^2, \quad I_2 = \frac{1}{2} M(R^2 + r^2)$$

(∵ they have the same mass)

Assuming the same (frictional) torque, $\tau$, acts on both the wheels,

$$\tau = I_1 \alpha_1 = I_2 \alpha_2$$

Since $I_1 > I_2$, \hspace{0.2cm} $\alpha_1 > \alpha_2$.

$$\omega = \omega_0 + \alpha t$$

Since the final angular velocity $\omega = 0$,

$$\alpha_1 = -\frac{\omega_0}{t_1} \quad \text{and} \quad \alpha_2 = -\frac{\omega_0}{t_2}$$

\therefore \hspace{0.2cm} $\frac{\omega_0}{t_1} > \frac{\omega_0}{t_2}$

\therefore \hspace{0.2cm} $t_1 < t_2$

---

Solved Problems 1.9

Q. 121. Solve the following:

1. A torque of magnitude 400 N·m, acting on a body of mass 40 kg, produces an angular acceleration of 20 rad/s$^2$. Calculate the moment of inertia and radius of gyration of the body. (3 marks)

**Solution:**

**Data:** $\tau = 400$ N·m, $M = 40$ kg, $\alpha = 20$ rad/s$^2$

The moment of inertia,

$$I = \frac{\tau}{\alpha} = \frac{400}{20} = 20 \text{ kg·m}^2$$

If $k$ is the radius of gyration of the body,

$$I = Mk^2$$

\therefore \hspace{0.2cm} $k^2 = \frac{I}{M} = \frac{20}{40} = 0.5$

\therefore \hspace{0.2cm} $k = \sqrt{0.5} = 0.7071 \text{ m}$

---

2. A body starts rotating from rest. Due to a couple of 20 N·m, it completes 60 revolutions in one minute. Find the moment of inertia of the body. (3 marks)

**Solution:**

**Data:** $\omega_0 = 0$, $\tau = 20$ N·m, $t = 1$ min = 60 s, $\theta = 60$ rev

$$\theta = \omega_0 t + \frac{1}{2} \alpha t^2$$

\therefore \hspace{0.2cm} $120 \pi = 0 + \frac{1}{2} \alpha (60 \times 60)$

\therefore \hspace{0.2cm} $\alpha = \frac{120 \pi \times 2}{60 \times 60} = \frac{\pi}{15} \text{ rad/s}^2$

$$\tau = I \alpha$$

The moment of inertia of the body,

$$I = \frac{\tau}{\alpha} = \frac{20}{\pi/15} = \frac{300}{\pi} = \frac{300}{3.142} = 95.48 \text{ kg·m}^2$$

---

3. A wheel of moment of inertia 2 kg·m$^2$ rotates at 50 rpm about its transverse axis. Find the torque that can stop the wheel in one minute. (3 marks)

**Solution:**

**Data:** $I = 2$ kg·m$^2$, $f = 50$ rpm = $\frac{50}{60} = \frac{5}{6} \text{ rev/s}$, $t = 60$ s

**Data:** $I = 2$ kg·m$^2$, $f = 50$ rpm = $\frac{50}{60} = \frac{5}{6} \text{ rev/s}$, $t = 60$ s
\[ \alpha = \frac{\omega - \omega_0}{t} = \frac{0 - 2\pi f_0}{t} \]
\[ = \frac{-2\pi(\frac{\pi}{60})}{60} = -\frac{\pi}{36} \, \text{rad/s}^2 \]

The magnitude of the required torque,
\[ \tau = I|\alpha| = 2\left(\frac{\pi}{36}\right) = \frac{3.142}{18} = 0.1746 \, \text{N-m} \]

(4) A circular disc of moment of inertia 10 kg·m² is rotated about its transverse symmetry axis at a constant frequency of 60 rpm by an electric motor of power 31.42 watts. When the motor is switched off, how many rotations does it complete before coming to rest? (3 marks)

Solution:
Data: \( I = 10 \, \text{kg·m}^2, \, P = 31.42 \, \text{watts} \),
\( f = 60 \, \text{rpm} = 60/60 \, \text{Hz} = 1 \, \text{Hz} \)

In rotational motion,
\[ \text{power} = \tau \times \omega \]
\[ \therefore \, P = \tau \omega \]
\[ \therefore \, \tau = \frac{P}{\omega} = \frac{P}{2\pi f} = \frac{3.142}{2\pi \times 1} = 5 \, \text{N-m} \]

This torque provided by the motor overcomes the torque of the frictional forces and maintains a constant frequency of rotation.

When the motor is switched off, the disc slows down due to the retarding torque of the frictional forces.

\[ \therefore \, \text{Retarding torque}, \, \tau_r = -5 \, \text{N-m} \]
\[ \therefore \, \text{Retardation}, \, \alpha = \frac{\tau}{I} = \frac{-5}{10} = -0.5 \, \text{rad/s}^2 \]

In this case, \( f_1 = 1 \, \text{Hz} \) and \( f_2 = 0 \)
\[ \omega_2^2 = \omega_1^2 + 2\alpha \theta \]
\[ \therefore \, (2\pi f_2)^2 = (2\pi f_1)^2 + 2\alpha \theta \]
\[ \therefore \, 0 = 4\pi^2 + 2 \times (-0.5) \times \theta \]
\[ \therefore \, \theta = 4\pi^2 \, \text{rad} \]

The number of rotations completed by the disc before coming to rest is \[ \frac{\theta}{2\pi} = \frac{4\pi^2}{2\pi} = 2\pi \times 3.142 \]
\[ = 6.284 \, \text{rotations} \]

(5) A flywheel of mass 4 kg and radius 10 cm, rotating with a uniform angular velocity of 5 rad/s, is subjected to a torque of 0.01 N·m for 10 seconds. If the torque increases the speed of rotation, find
(i) the final angular velocity of the flywheel
(ii) the change in its angular velocity
(iii) the change in its angular momentum
(iv) the change in its kinetic energy. (1 mark each)

Solution:
Data: \( M = 4 \, \text{kg}, \, R = 10 \, \text{cm} = 0.1 \, \text{m}, \, \omega_1 = 5 \, \text{rad/s}, \, \tau = 0.01 \, \text{N-m}, \, t = 10 \, \text{s} \)
\[ I = \frac{MR^2}{2} = \frac{4 \times 0.01}{2} = 0.02 \, \text{kg·m}^2 \]
\[ \alpha = \frac{\tau}{I} = \frac{0.01}{0.02} = 0.5 \, \text{rad/s} \]

(i) The final angular velocity of the flywheel,
\[ \omega_2 = \omega_1 + \alpha \]
\[ = 5 + 0.5 \times 10 = 10 \, \text{rad/s} \]

(ii) The change in its angular velocity
\[ = \omega_2 - \omega_1 = 5 \, \text{rad/s} \]

(iii) The change in its angular momentum
\[ = I\omega_2 - I\omega_1 = I(\omega_2 - \omega_1) \]
\[ = 0.02 \times 5 = 0.1 \, \text{kg·m}^2/\text{s} \]

(iv) The change in its kinetic energy
\[ = \frac{1}{2} I\omega_2^2 - \frac{1}{2} I\omega_1^2 = \frac{1}{2} I(\omega_2^2 - \omega_1^2) \]
\[ = \frac{1}{2} \times 0.02 \times [(10)^2 - (5)^2] = 0.75 \, \text{J} \]

(6) A torque of 20 N·m sets a stationary circular disc into rotation about a transverse axis through its centre and acts for 2π seconds. If the disc has a mass 10 kg and radius 0.2 m, what is its frequency of rotation after 2π seconds? (3 marks)

Solution:
Data: \( \tau = 20 \, \text{N-m}, \, t = 2\pi \, \text{s}, \, M = 10 \, \text{kg}, \, R = 0.1 \, \text{m} \)
Let \( f_1 \) and \( f_2 \) be the initial and final frequencies of rotation of the disc, and \( \omega_1 \) and \( \omega_2 \) be its initial and final angular speeds. Since the disc was initially stationary, \( f_1 = \omega_1 = 0 \) and \( \omega_2 = 2\pi f_2 \).

The MI of the disc about the given axis is
\[ I = \frac{MR^2}{2} = \frac{10 \times (0.2)^2}{2} = 0.2 \, \text{kg·m}^2 \]

Torque, \( \tau = I\alpha \)

Angular acceleration, \( \alpha = \frac{\tau}{I} = \frac{20}{0.2} = 100 \, \text{rad/s}^2 \)
Now, \( \omega_2 = \omega_1 + \Delta t = 0 + \Delta t \)
\[
\therefore 2\pi f_2 = \Delta t
\]
\[
\therefore f_2 = \frac{\Delta t}{2\pi} = \frac{100(2\pi)}{2\pi} = 100 \text{ Hz}
\]

(7) A rope is wound around a hollow cylinder of mass 3 kg and radius 40 cm. If the rope is pulled downwards with a force of 30 N, find (i) the angular acceleration of the cylinder (ii) the linear acceleration of the rope. (3 marks)

Solution:

Data: \( M = 3 \text{ kg}, R = 0.4 \text{ m}, F = 30 \text{ N} \)

(i) The MI of a hollow cylinder about its cylinder axis, \( I = MR^2 \)

The torque on the cylinder, \( \tau = RF \)

Also, \( \tau = I \alpha = MR^2 \alpha \)

\( \therefore RF = MR^2 \alpha \)

The angular acceleration,
\[
\alpha = \frac{RF}{MR^2} = \frac{F}{MR} = \frac{30}{(3)(0.4)} = 25 \text{ rad/s}^2
\]

(ii) The linear acceleration of the rope
= the tangential acceleration \( a_t \)
\[
a_t = R \alpha = 25 \times 0.4 = 10 \text{ m/s}^2
\]

Unit

1.10 Conservation of angular momentum

Q. 122. State and prove the principle (or law) of conservation of angular momentum. (3 marks)

Ans. Principle (or law) of conservation of angular momentum: The angular momentum of a body is conserved if the resultant external torque on the body is zero.

Proof: Consider a moving particle of mass \( m \) whose position vector with respect to the origin at any instant is \( \vec{r} \).

Then, at this instant, the linear velocity of this particle is \( \vec{v} = \frac{d\vec{r}}{dt} \), its linear momentum is \( \vec{p} = m\vec{v} \) and its angular momentum about an axis through the origin is \( \vec{L} = \vec{r} \times \vec{p} \).

Suppose its angular momentum \( \vec{L} \) changes with time due to a torque \( \vec{\tau} \) exerted on the particle.

The time rate of change of its angular momentum,
\[
\frac{d\vec{L}}{dt} = \frac{d}{dt} (r \times p) = \frac{d\vec{r}}{dt} \times p + r \times \frac{dp}{dt}
\]
\[
= \vec{v} \times m\vec{v} + r \times \frac{dp}{dt}
\]
\[
= \vec{r} \times \vec{F} \quad (\because \vec{v} \times \vec{v} = 0)
\]
where \( \frac{dp}{dt} = \vec{F} \), the net force on the particle.

Hence, if \( \vec{\tau} = 0 \), \( \frac{d\vec{L}}{dt} = 0 \)

\( \therefore \vec{L} \) = constant, i.e., \( \vec{L} \) is conserved. This proves the principle (or law) of conservation of angular momentum.

Alternate Proof: Consider a rigid body rotating with angular acceleration \( \vec{\alpha} \) about the axis of rotation. If \( I \) is the moment of inertia of the body about the axis of rotation, \( \vec{\omega} \) the angular velocity of the body at time \( t \) and \( \vec{L} \) the corresponding angular momentum of the body, then
\[
\vec{L} = I\vec{\omega} \quad \cdots (1)
\]

Differentiating this with respect to \( t \), we get:

Rate of change of angular momentum with time,
\[
\frac{d\vec{L}}{dt} = \frac{d}{dt} (I\vec{\omega}) = I \frac{d\vec{\omega}}{dt}
\]
\[
(\because I = \text{constant in a particular case, i.e., about the given axis of rotation})
\]
\[
\therefore \frac{d\vec{L}}{dt} = I \vec{\alpha} \quad (\because \vec{\alpha} = \frac{d\vec{\omega}}{dt}) \quad \cdots (2)
\]

But \( \vec{\tau}_{\text{ext}} = I \vec{\alpha} \), where \( \vec{\tau}_{\text{ext}} \) is the resultant external torque on the body.

\( \therefore \vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt} \)

Hence, if \( \vec{\tau}_{\text{ext}} = 0 \), \( \frac{d\vec{L}}{dt} = 0 \)

\( \therefore \vec{L} \) = constant, i.e., \( \vec{L} \) is conserved. This proves the principle (or law) of conservation of angular momentum.

Q. 123. State the law (or principle) of conservation of angular momentum and explain it with a suitable example. (3 marks)

OR

★ State and explain the principle of conservation of angular momentum. Use a suitable illustration.

Do we use it in our daily life? When? (3 marks)
Law (or principle) of conservation of angular momentum: The angular momentum of a body is conserved if the resultant external torque on the body is zero.

Explanation: This law (or principle) is used by a figure skater or a ballerina to increase their speed of rotation for a spin by reducing the body’s moment of inertia. A diver too uses it during a somersault for the same reason.

Ice dance: Twizzle and spin are elements of the sport of figure skating. In a twizzle a skater turns several revolutions while travelling on the ice. In a dance spin, the skater rotates on the ice skate and centred on a single point on the ice. The torque due to friction between the ice skate and the ice is small. Consequently, the angular momentum of a figure skater remains nearly constant.

For a twizzle of smaller radius, a figure skater draws her limbs close to her body to reduce moment of inertia and increase frequency of rotation. For larger rounds, she stretches out her limbs to increase moment of inertia which reduces the angular and linear speeds.

A figure skater usually starts a dance spin in a crouch, rotating on one skate with the other leg and both arms extended. She rotates relatively slowly because her moment of inertia is large. She then slowly stands up, pulling the extended leg and arms to her body. As she does so, her moment of inertia about the axis of rotation decreases considerably and thereby her angular velocity substantially increases to conserve angular momentum.

Q. 124. A diver pulls her limbs in and curls up her body for a somersault in flight but extends her limbs just before entering the water. Explain the effect of both actions on her angular velocities. Also explain the effect on her angular momentum.

Ans. Refer to the example (2) in the answer to Q. 123.

Q. 125. What happens when a ballet dancer stretches her arms while taking turns?

Ans. When a ballet dancer stretches her arms while pirouetting, her moment of inertia increases, and consequently her angular speed decreases to conserve angular momentum.

Q. 126. If the Earth suddenly shrinks so as to reduce its volume, mass remaining unchanged, what will be the effect on the duration of the day?

Ans. If the Earth suddenly shrinks, mass remaining constant, the moment of inertia of the Earth will decrease, and consequently the angular velocity of rotation $\omega$ about its axis will increase. Since period $T \propto \frac{1}{\omega}$, the duration of the day $T$ will decrease.

Q. 127. Two discs of moments of inertia $I_1$ and $I_2$ about their transverse symmetry axes, respectively rotating with angular velocities $\omega_1$ and $\omega_2$, are brought into contact with their rotation axes coincident. Find the angular velocity of the composite disc.

Ans. Refer to the answer to Q. 124.
Ans. We assume that the initial angular momenta ($\vec{L}_1$ and $\vec{L}_2$) of the discs are either in the same direction or in opposite directions. Then,

the total initial angular momentum

$$\vec{L} = \vec{L}_1 + \vec{L}_2 = I_1\omega_1 + I_2\omega_2$$

After they are coupled, the total moment of inertia, i.e., the moment of inertia of the composite disc is $I = I_1 + I_2$ and the common angular velocity is $\omega$. Assuming conservation of angular momentum,

$$I\omega = (I_1 + I_2)\omega = I_1\omega_1 + I_2\omega_2$$

$$\therefore \omega = \frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}$$

If $\omega_1$ and $\omega_2$ are in the same direction, $\omega = \frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}$. If $\omega_1$ and $\omega_2$ are in opposite directions, $\omega = \left|\frac{I_1\omega_1 - I_2\omega_2}{I_1 + I_2}\right|$.

Q. 128. A boy standing at the centre of a turntable with his arms outstretched is set into rotation with angular speed $\omega$ rev/min. When the boy folds his arms back, his moment of inertia reduces to $\frac{2}{5}$th its initial value. Find the ratio of his final kinetic energy of rotation to his initial kinetic energy.

(2 marks)

Ans. Data: $I_2 = \frac{2}{5} I_1$

$$L = I\omega$$

Assuming the angular momentum $\vec{L}$ is conserved, in magnitude,

$$I_1\omega_1 = I_2\omega_2$$

$$\therefore \frac{\omega_2}{\omega_1} = \frac{I_1}{I_2} = \frac{5}{2}$$

Rotational KE, $E = \frac{1}{2} I\omega^2$

$$E_2 = \frac{I_2}{I_1}\left(\frac{\omega_2}{\omega_1}\right)^2 = \frac{2}{5}\left(\frac{2}{5}\right)^2 = \frac{5}{2}$$

This gives the required ratio.

Q. 129. Name the quantity that is conserved when

(i) $\vec{F}_{\text{external}}$ is zero (ii) $\vec{\tau}_{\text{external}}$ is zero. (1 mark)

Ans. (i) Total linear momentum is conserved when $\vec{F}_{\text{external}}$ is zero.

(ii) Angular momentum is conserved when $\vec{\tau}_{\text{external}}$ is zero.

Q. 130. What is the rotational analogue of the equation

$$\vec{F}_{\text{external}} = \frac{d\vec{p}}{dt}.$$  

(1 mark)

Ans. $\vec{\tau}_{\text{external}} = \frac{d\vec{L}}{dt}$. 

Q. 131. Fly wheels used in automobiles and steam engines producing rotational motion have discs with a large moment of inertia. Explain why?

(2 marks)

Ans. A flywheel is used as (i) a mechanical energy storage, the energy being stored in the form of rotational kinetic energy (ii) a direction and speed stabilizer. A flywheel rotor is typically in the form of a disc. Rotational kinetic energy, $E_{\text{rot}} = \frac{1}{2} I\omega^2$, where $I$ is the moment of inertia and $\omega$ is the angular speed. That is, $E_{\text{rot}} \propto I$. Therefore, higher the moment of inertia, the higher is the rotational kinetic energy that can be stored or recovered.

Also, angular momentum, $\vec{L} = I\vec{\omega}$, i.e., $|\vec{L}| \propto I$. A torque aligned with the symmetry axis of a flywheel can change its angular velocity and thereby its angular momentum. A flywheel with a large angular momentum will require a greater torque to change its angular velocity. Thus, a flywheel can be used to stabilize direction and magnitude of its angular velocity by undesired torques.

Solved Problems 1.10

Q. 132. Solve the following:

(1) A uniform horizontal disc is freely rotating about a vertical axis passing through its centre at the rate of 180 rpm. A blob of wax of mass 1.9 g falls on it and sticks to it at 25 cm from the axis. If the frequency of rotation is reduced by 60 rpm, calculate the moment of inertia of the disc. (3 marks)

Solution:

Data: $f_1 = 180 \text{ rpm} = 180/60 \text{ rot/s} = 3 \text{ rot/s}$,

$f_2 = (180 - 60) \text{ rpm} = 120/60 \text{ rot/s} = 2 \text{ rot/s}$,

$m = 1.9 \text{ g} = 1.9 \times 10^{-3} \text{ kg}$, $r = 25 \text{ cm} = 0.25 \text{ m}$

Let $I_1$ be the MI of the disc. Let $I_2$ be the MI of the disc and the blob.
(2) A horizontal disc is rotating about a transverse axis through its centre at 100 rpm. A 20 gram blob of wax falls on the disc and sticks to it at 5 cm from its axis. The moment of inertia of the disc about its axis through its centre is $2 \times 10^{-4}$ kg·m$^2$. Calculate the new frequency of rotation of the disc.

**Solution**:

**Data**: $f_1 = 100$ rpm, $m = 20$ g = $20 \times 10^{-3}$ kg, $r = 5$ cm = $5 \times 10^{-2}$ m, $I_1 = I_{\text{disc}} = 2 \times 10^{-4}$ kg·m$^2$

The MI of the disc and blob of wax is

$I_2 = I_1 + mr^2$

$= (2 \times 10^{-4}) + (20 \times 10^{-3})(5 \times 10^{-2})^2$

$= (2 \times 10^{-4}) + (20 \times 10^{-3})(25 \times 10^{-4})$

$= (2 + 0.5) \times 10^{-4} = 2.5 \times 10^{-4}$ kg·m$^2$

By the principle of conservation of angular momentum, $I_1 \omega_1 = I_2 \omega_2$.

$\therefore I_1 (2\pi f_1) = I_2 (2\pi f_2)$

$\therefore f_2 = \frac{I_1 f_1}{I_2} = \frac{(2 \times 10^{-4})(100)}{2.5 \times 10^{-4}} = \frac{200}{5/2} = 80$ rpm

This is the new frequency of rotation.

(3) A potter’s wheel is set into rotation at 100 rpm. It is in the form of a disc of mass 10 kg and radius 0.4 m. A lump of clay (to be treated as a particle) of mass 1.6 kg falls and adheres to the wheel at a distance $x$ from its centre. Calculate $x$ if the wheel now rotates at 80 rpm.

**Solution**:

**Data**: $f_1 = 100$ rpm, $f_2 = 80$ rpm, $M = 10$ kg, $R = 0.4$ m, $m = 1.6$ kg

$I_1 = I_{\text{wheel}} = \frac{1}{2} MR^2 = \frac{1}{2}(10)(0.4)^2 = 0.8$ kg·m$^2$

The MI of the wheel and the lump of clay is

$I_2 = I_{\text{wheel}} + mx^2$

By the principle of conservation of angular momentum, $I_1 \omega_1 = I_2 \omega_2$.

$\therefore I_1 (2\pi f_1) = I_2 (2\pi f_2)$

$\therefore f_2 = \frac{I_1 f_1}{I_2} = \frac{f_1}{f_2} I_{\text{wheel}}$

$\therefore mx^2 = \left(\frac{f_1}{f_2} - 1\right) I_{\text{wheel}} = \left(\frac{100}{80} - 1\right)(0.8)$

$= \left(\frac{5}{4} - 1\right)(0.8) = 0.2$ kg·m$^2$

$\therefore x^2 = \frac{0.2}{1.6} = \frac{1}{8} \therefore x = \frac{1}{\sqrt{8}} m = 0.3536$ m

(4) A ballet dancer spins about a vertical axis at $2.5 \pi$ rad/s with his arms outstretched. With the arms folded, the MI about the same axis of rotation changes by 25%. Calculate the new speed of rotation in rpm.

**Solution**:

Let $I_1$, $\omega_1$ and $f_1$ be the moment of inertia, angular velocity and frequency of rotation of the ballet dancer with arms outstretched, and $I_2$, $\omega_2$ and $f_2$ be the corresponding quantities with arms folded.

**Data**: $\omega_1 = 2.5 \pi$ rad/s

Since moment of inertia with arms folded is less than that with arms outstretched,

$I_2 < I_1$

$\therefore I_2 = I_1 - 0.25 I_1 = 0.75 I_1 = \frac{3}{4} I_1$

$\omega_2 = 2\pi f_2 = 2.5 \pi$

$\therefore f_2 = \frac{2.5 \pi}{2\pi} = \frac{5}{4}$ Hz

According to the principle of conservation of angular momentum, $I_1 \omega_1 = I_2 \omega_2$.

$\therefore I_1 (2\pi f_1) = I_2 (2\pi f_2)$

The new frequency of rotation is

$f_2 = \frac{I_1 f_1}{I_2} = \frac{4}{3} \times \frac{5}{4} = \frac{5}{3} = 1.6667$ Hz

$= \frac{5}{3} \times 60$ rpm

$= 100$ rpm

(5) Two wheels, each of moment of inertia 4 kg·m$^2$, rotate side by side at the rate of 120 rpm and 240 rpm in opposite directions. If both the wheels
are coupled by a weightless shaft so that they now rotate with a common angular speed, find this new rate of rotation.  

(3 marks)

Solution:

Data: \( I = 4 \ \text{kg-m}^2, f_1 = 120 \ \text{rpm}, f_2 = 240 \ \text{rpm} \)

Initially, the angular velocities of the two wheels (\( \omega_1 \) and \( \omega_2 \)) and, therefore, their angular momenta (\( L_1 \) and \( L_2 \)) are in opposite directions.

The magnitude of the total initial angular momentum is

\[ L_1 + L_2 = -I\omega_1 + I\omega_2 \]  

\[ = 2\pi I (f_2 - f_1) \]  

\[ = 2\pi I \left( \frac{240 - 120}{2} \right) \]  

\[ = 60 \ \text{rpm} \]

This gives their new rate of rotation.

(6) A homogeneous (uniform) rod XY of length \( L \) and mass \( M \) is pivoted at the centre C such that it can rotate freely in a vertical plane. Initially, the rod is horizontal. A blob of wax of the same mass \( M \) as that of the rod falls vertically with speed \( V \) and sticks to the rod midway between points C and Y.

As a result, the rod rotates with angular speed \( \omega \). What will be the angular speed in terms of \( V \) and \( L \)?  

(3 marks)

Solution: The initial angular momentum of the rod is zero.

The initial angular momentum of the falling blob of wax about the point C is (in magnitude)

\[ = \text{mass} \times \text{speed} \times \text{perpendicular distance between its direction of motion and point C} \]

\[ = M V \left( \frac{L}{4} \right) \]

The total initial angular momentum of the rod and blob of wax \( = M V L / 4 \)  

\[ \ldots \ (1) \]

Fig. 1.58

After the blob of wax sticks to the rod, and the system rotates with an angular speed \( \omega \) about the horizontal axis through point C perpendicular to the plane of the figure, the total final angular momentum of the system about this axis

\[ = \left( \frac{ML^2}{12} + M \left( \frac{L}{4} \right)^2 \right) \omega \]

\[ = \left( \frac{ML^2}{12} + \frac{ML^2}{16} \right) \omega = \frac{7ML^2}{48} \cdot \omega \]  

\[ \ldots \ (2) \]

From Eqs (1) and (2), by the principle of conservation of angular momentum,

\[ \frac{7ML^2}{48} \cdot \omega = \frac{MVL}{4} \]

\[ \therefore \omega = \frac{12V}{7L} \]

This gives the required angular speed.

(7) A satellite moves around the Earth in an elliptical orbit such that at perigee (closest approach) it is two Earth radii above the Earth’s surface. At apogee (farthest position), it travels with one-fourth the speed it has at perigee. In terms of the Earth’s radius \( R \), what is the maximum distance of the satellite from the Earth’s surface?  

(2 marks)

Solution: Let \( r_p \) and \( r_a \) be the distances of the satellite from the centre of the Earth at perigee and apogee, respectively. Let \( v_p \) and \( v_a \) be its linear (tangential) velocities at perigee and apogee.

Data: \( r_p = 2R + R = 3R, v_a = \frac{1}{4} v_p \)

Let \( L_p \) and \( L_a \) be the angular momenta of the satellite about the Earth’s centre. Because the gravitational force \( \vec{F} \) on the satellite due to the Earth is always radially towards the centre of the Earth, its direction is opposite to that of the position vector \( \vec{r} \) of the satellite relative to the centre of the Earth, so that the torque \( \vec{\tau} = \vec{r} \times \vec{F} = 0 \). Hence, the angular momentum of the satellite about the Earth’s centre is...
constant in time.

\[ L_p = L_a \]

If \( m \) is the mass of the satellite,

\[ mv_p^2 = mv_a^2 \]

\[ r_p = \frac{v_p}{v_a} \]

\[ r_a = \frac{3R}{v_p} \]

\[ r_a = 4(3R) = 12R \]

At apogee, the distance of the satellite from the Earth’s surface is \( 12R \).

(8) A torque of 100 N·m is applied to a body capable of rotating about a given axis. If the body starts from rest and acquires kinetic energy of 10000 J in 10 seconds, find (i) its moment of inertia about the given axis (ii) its angular momentum at the end of 10 seconds. (3 marks)

Solution :

Data :

\( \tau = 100 \text{ N·m}, \omega_i = 0, E_i = 0, E_f = 10^4 \text{ J}, t = 10 \text{ s} \)

\[ \tau = \Delta L \]

Since the body starts from rest, its initial angular momentum, \( L_i = 0 \).

The final angular momentum,

\[ L_f = \tau \Delta t = (100)(10) = 10^3 \text{ kg·m}^2/\text{s} \]

The final rotational kinetic energy,

\[ E_f = \frac{1}{2} I_f \omega_f \]

\[ \omega_f = \frac{2E_i}{L_f} \]

\[ = \frac{2 \times 10^4}{10^3} = 20 \text{ rad/s} \]

The moment of inertia of the body,

\[ I = \frac{L_i}{\omega_i} \]

\[ = \frac{10^3}{20} = 50 \text{ kg·m}^2 \]

(9) Two identical metal beads, each of mass \( M \) but negligible width, can slide along a thin smooth uniform horizontal rod of mass \( M \) and length \( L \). The rod is capable of rotating about a vertical axis passing through its centre. Initially, the beads are almost touching the axis of rotation and the rod is rotating at speed of 14 rad/s. Find the angular speed of the system when the beads have moved up to the ends of the rod. (Assume that no external torque acts on the system.) (3 marks)

Solution :

Data : \( \omega_i = 14 \text{ rad/s} \)

The MI of the rod about a transverse axis through its CM,

\[ I_{rod} = \frac{ML^2}{12} \]

Since the beads are almost particle-like, and initially touching the rotation axis, their MI about the vertical axis is taken to be zero.

When the beads move up to the ends of the rod, \( r = L/2 \), their MI about the vertical axis is

\[ I_{beads} = 2Mr^2 = 2M \left( \frac{L}{2} \right)^2 = \frac{ML^2}{2} \]

\[ I_1 = I_{rod} = \frac{ML^2}{12} \]

and

\[ I_2 = I_{rod} + I_{beads} = \frac{ML^2}{12} + \frac{ML^2}{2} = \frac{7ML^2}{12} \]

By the principle of conservation of angular momentum,

\[ I_2 \omega_2 = I_1 \omega_1 \]

The final angular speed of the system,

\[ \omega_2 = \frac{I_1 \omega_1}{I_2} = \frac{\left( \frac{ML^2}{12} \right)(14)}{\left( \frac{7ML^2}{12} \right)} = 2 \text{ rad/s} \]

Unit

1.11 Rolling motion

1.11.1 Linear acceleration and speed of a body rolling down an inclined plane

Q. 133. Discuss how pure rolling (i.e., rolling without slipping) on a plane surface is a combined translational and rotational motion. (3 marks)

Ans. Rolling motion (without slipping) is an important case of combined translation and rotation. Consider a circularly symmetric rigid body, like a wheel or a disc, rolling on a plane surface with friction along a straight path.
The centre of mass of the wheel is at its geometric centre O. For purely translational motion (the wheel sliding smoothly along the surface without rotating at all), every point on the wheel has the same linear velocity \( \vec{v} = \vec{v}_O \) as the centre O. For purely rotational motion (as if the horizontal rotation axis through O were stationary), every point on the wheel rotates about the axis with angular velocity \( \omega \); in this case, every point on the rim has the same linear speed \( \omega R \).

We view the combined motion in the inertial frame of reference in which the surface is at rest. In this frame, since there is no slipping, the point of contact of the wheel with the surface is instantaneously stationary, \( v_A = 0 \), so that the wheel is turning about an instantaneous axis through the point of contact A. The instantaneous linear speed of point C (at the top) is \( v_C = \omega (2R) \) – faster than any other point of the wheel.

Q. 134. Deduce an expression for the kinetic energy of a body rolling on a plane surface without slipping.  
(2 marks)  
OR  
Obtain an expression for the total kinetic energy of a rolling body in the form \(\frac{1}{2} Mv^2 \left[ 1 + \frac{k^2}{R^2} \right] \).  
(2 marks)  
OR  
Derive an expression for the kinetic energy when a rigid body is rolling on a horizontal surface without slipping. Hence, find the kinetic energy of a solid sphere.  
(3 marks)  

Ans. Consider a symmetric rigid body, like a sphere or a wheel or a disc, rolling on a plane surface with friction along a straight path. Its centre of mass (CM) moves in a straight line and, if the frictional force on the body is large enough, the body rolls without slipping. Thus, the rolling motion of the body can be treated as translation of the CM and rotation about an axis through the CM. Hence, the kinetic energy of a rolling body is

\[ E = E_{\text{tran}} + E_{\text{rot}} \]  
\[ = \frac{1}{2} M v^2 + \frac{1}{2} I \omega^2 \]  
\[ = \frac{1}{2} M v^2 \left( 1 + \frac{1}{Mk^2} \right) \]  
\[ = \frac{1}{2} M v^2 \left( 1 + \frac{k^2}{MR^2} \right) \]  
\[ = \frac{1}{2} M \omega^2 R^2 \left( 1 + \frac{k^2}{R^2} \right) \]

Also, from Eq. (2),

\[ E = \frac{1}{2} M \omega^2 R^2 \left( 1 + \frac{k^2}{R^2} \right) = \frac{1}{2} M \omega^2 (R^2 + k^2) \]
Q. 135. Discuss the interlink between translational, rotational and total kinetic energies of a rigid object that rolls without slipping. (1 mark)

Ans. Refer to the answer to Q. 134 up to Eq. (5).

Q. 136. A uniform solid sphere of mass 10 kg rolls on a horizontal surface. If its linear speed is 2 m/s, what is its total kinetic energy? (1 mark)

Ans. Total kinetic energy of the sphere

\[ E = \frac{1}{2} M v^2 = \frac{7}{10} \times 10 \times (2)^2 = 28 \, \text{J} \]

Q. 137. A disc of mass 4 kg rolls on a horizontal surface. If its linear speed is 3 m/s, what is its total kinetic energy? (1 mark)

Ans. Total kinetic energy of the disc

\[ E = \frac{3}{4} M v^2 = \frac{3}{4} \times 4 \times (3)^2 = 27 \, \text{J} \]

Q. 138. Assuming the expression for the kinetic energy of a body rolling on a plane surface without slipping, deduce the expression for the total kinetic energy of rolling motion for (i) a ring (ii) a disk (iii) a hollow sphere. Also, find the ratio of rotational kinetic energy to total kinetic energy for each body. (2 marks each)

Ans. For a body of mass \( M \) and radius of gyration \( k \), rolling on a plane surface without slipping with speed \( v \), its total KE and rotational KE are respectively

\[ E = \frac{1}{2} M v^2 (1 + \beta) \]

and \( E_{\text{rot}} = \frac{1}{2} I \omega^2 = \frac{1}{2} (Mk^2) \left( \frac{v^2}{R^2} \right) = \frac{1}{2} \beta M v^2 \]

where \( \beta = \frac{1}{MR^2} = \frac{k^2}{R^2} \).

(i) For a ring, \( I = MR^2 \), so that \( \beta = 1 \).

\[ \therefore E = \frac{1}{2} M v^2 (1 + 1) = Mv^2 \]

and \( E_{\text{rot}} = \frac{1}{2} M v^2 \therefore E_{\text{rot}}/E = \frac{1}{2} \)

(ii) For a disc, \( I = \frac{1}{2} MR^2 \), so that \( \beta = \frac{1}{2} \).

\[ \therefore E = \frac{1}{2} M v^2 \left( 1 + \frac{1}{2} \right) = \frac{3}{4} M v^2 \]

and \( E_{\text{rot}} = \frac{1}{4} M v^2 \therefore E_{\text{rot}}/E = \frac{1}{3} \)

(iii) For a spherical shell (hollow sphere), \( I = \frac{2}{3} MR^2 \), so that \( \beta = \frac{2}{3} \).

\[ \therefore E = \frac{1}{2} M v^2 \left( 1 + \frac{2}{3} \right) = \frac{5}{6} M v^2 \]

and \( E_{\text{rot}} = \frac{1}{3} M v^2 \quad \text{and} \quad E_{\text{rot}}/E = \frac{2}{5} \)

[Note: The moment of inertia of all the round bodies above can be expressed as \( I = \beta MR^2 \), where \( \beta \) is a pure number less than or equal to 1. \( \beta \) is equal to 1 for a ring or a thin-walled hollow cylinder, \( \frac{1}{2} \) for a disc or solid cylinder, \( \frac{2}{3} \) for a hollow sphere and \( \frac{2}{5} \) for a solid sphere. All uniform rings or hollow cylinders of the same mass and moving with the same speed have the same total kinetic energy, even if their radii are different. All discs or solid cylinders of the same mass and moving with the same speed have the same total kinetic energy; all solid spheres of the same mass and moving with the same speed have the same total kinetic energy. Also, for the same mass and speed, bodies with small \( c \) have less total kinetic energy.]

Q. 139. A circularly symmetric rigid body starts from rest and rolls down a plane inclined at an angle \( \theta \) to the horizontal without slipping. Derive expressions for (i) its speed at the bottom of the plane (ii) its acceleration (iii) the time it takes to reach the bottom. (4 marks) OR

* A rigid body rolls down an inclined plane without slipping. Derive the expressions for the acceleration along the plane and the speed after falling through a certain vertical distance. (3 marks)

Ans. Consider a circularly symmetric rigid body, like a sphere or a wheel or a disc, rolling with friction
down a plane inclined at an angle $\theta$ to the horizontal. If the frictional force on the body is large enough, the body rolls without slipping.

Let $M$ and $R$ be the mass and radius of the body. Let $I$ be the moment of inertia of the body for rotation about an axis through its centre. Let the body start from rest at the top of the incline at a height $h$. Let $v$ be the translational speed of the centre of mass at the bottom of the incline. Then, its kinetic energy at the bottom of the incline is

$$E = \frac{1}{2} Mv^2 \left[ 1 + \frac{I}{MR^2} \right] = \frac{1}{2} Mv^2 (1 + \beta) \quad \ldots (1)$$

where $\beta = \frac{I}{MR^2}$.

If $k$ is the radius of gyration of the body,

$$I = Mk^2 \quad \text{and} \quad \beta = \frac{k^2}{R^2}$$

From conservation of energy,

$$(KE + PE)_{\text{initial}} = (KE + PE)_{\text{final}} \quad \ldots (2)$$

$$\therefore \quad 0 + Mgh = \frac{1}{2} Mv^2 (1 + \beta) + 0$$

$$\therefore \quad Mgh = \frac{1}{2} Mv^2 (1 + \beta) \quad \ldots (3)$$

$$\therefore \quad v^2 = \frac{2gh}{1 + \beta}$$

$$\therefore \quad v = \sqrt{\frac{2gh}{1 + \beta}} = \sqrt{\frac{2gh}{1 + (k^2/R^2)}} \quad \ldots (4)$$

Since $h = L \sin \theta$,

$$v = \sqrt{\frac{2gL \sin \theta}{1 + (k^2/R^2)}} \quad \ldots (5)$$

Let $a$ be the acceleration of the centre of mass of the body along the inclined plane. Since the body starts from rest,

$$v^2 = 2aL \quad \therefore a = \frac{v^2}{2L} \quad \ldots (6)$$

$$\therefore \quad a = \frac{2gL \sin \theta}{1 + \beta} \cdot \frac{1}{2L} = \frac{g \sin \theta}{1 + \beta} = \frac{g \sin \theta}{1 + (k^2/R^2)} \quad \ldots (7)$$

Starting from rest, if $t$ is the time taken to travel the distance $L$,

$$L = \frac{1}{2} at^2$$

$$\therefore \quad t = \sqrt{\frac{2L}{a}} = \sqrt{\frac{2L}{g \sin \theta \left( 1 + \frac{k^2}{R^2} \right)}} \quad \ldots (8)$$

[Note : For rolling without slipping, the contact point of the rigid body is instantaneously at rest relative to the surface of the inclined plane. Hence, the force of friction is static rather than kinetic, and does no work on the body. Thus, the force of static friction causes no decrease in the mechanical energy of the body and we can use the principle of conservation of energy.]

Q. 140. State the expression for the speed of a circularly symmetric body rolling without slipping down an inclined plane. Hence deduce the expressions for the speed of (i) a ring (ii) a solid cylinder (iii) a hollow sphere (iv) a solid sphere, having the same radii. (2 marks each)

Ans. Consider a circularly symmetric body, of mass $M$ and radius of gyration $k$, starting from rest on an inclined plane and rolling down without slipping. Its speed after rolling down through a height $h$ is

$$v = \sqrt{\frac{2gh}{1 + \beta}} = \sqrt{\frac{2gh}{1 + (k^2/R^2)}} \quad \ldots (1)$$

(i) Ring : $I = MR^2$, so that $\beta = 1$.

$$\therefore \quad v = \sqrt{\frac{2gh}{1 + 1}} = \sqrt{gh} \quad \ldots (2)$$

(ii) Solid cylinder or disc : $I = \frac{1}{2} MR^2$, so that $\beta = \frac{1}{2}$.

$$\therefore \quad v = \sqrt{\frac{2gh}{1 + \frac{1}{2}}} = \sqrt{\frac{4}{3} gh} \quad (= \sqrt{1.333gh}) \quad \ldots (3)$$

(iii) Spherical shell (hollow sphere) :

$$I = \frac{2}{3} MR^2$$, so that $\beta = \frac{2}{3}$.

$$\therefore \quad v = \sqrt{\frac{2gh}{1 + \frac{2}{3}}} = \sqrt{\frac{6}{5} gh} \quad (= \sqrt{1.2gh}) \quad \ldots (4)$$

(iv) Solid sphere : $I = \frac{2}{5} MR^2$, so that $\beta = \frac{2}{5}$.

$$\therefore \quad v = \sqrt{\frac{2gh}{1 + \frac{2}{5}}} = \sqrt{\frac{10}{7} gh} \quad (= \sqrt{1.428gh}) \quad \ldots (5)$$
Q. 141. State with reason if the statement is true or false : A wheel moving down a perfectly frictionless inclined plane will undergo slipping (not rolling) motion. (1 mark)

Ans. The statement is true.

Explanation : Rolling on a surface (horizontal or inclined) without slipping may be viewed as pure rotation about an horizontal axis through the point of contact, when viewed in the inertial frame of reference in which the surface is at rest. The point of contact of the wheel with the surface will be instantaneously at rest, resulting in a rolling motion, provided the wheel is able to ‘grip’ the surface, i.e., friction is necessary. With little or no friction, the wheel will slip at the point of contact. On an inclined plane, this will result in pure translation simultaneously at rest, resulting in a rolling motion.

Q. 142. A ring and a disc roll down an inclined plane through the same height. Compare their speeds at the bottom of the plane. (1 mark)

Ans. In the usual notation,
\[
\frac{v_{\text{disc}}}{v_{\text{ring}}} = \frac{\sqrt{(4/3)gh}}{\sqrt{gh}} = \frac{\sqrt{4}}{\sqrt{3}} = \frac{2}{\sqrt{3}}.
\]

Q. 143. State the expression for the acceleration of a circularly symmetric rigid body rolling without slipping down an inclined plane. Hence, deduce the acceleration of (i) a ring (ii) a solid cylinder (iii) a hollow cylinder (iv) a solid sphere, rolling without slipping down an inclined plane. (2 marks each)

Ans. A circularly symmetric rigid body, of radius \( R \) and radius of gyration \( k \), on rolling down an inclined plane of inclination \( \theta \) has an acceleration
\[
\alpha = \frac{g \sin \theta}{1 + \beta} = \frac{g \sin \theta}{1 + (k^2/R^2)}
\] ...

(i) Ring : \( I = MR^2 \), so that \( \beta = 1 \).
\[
\therefore \quad \alpha = \frac{g \sin \theta}{1 + 1} = \frac{1}{2} \frac{g \sin \theta}{g \sin \theta} = \frac{1}{2} \quad (= 0.5 g \sin \theta) \quad \text{... (2)}
\]

(ii) Solid cylinder or disc : \( I = \frac{1}{2} MR^2 \), so that \( \beta = \frac{1}{2} \).
\[
\therefore \quad \alpha = \frac{g \sin \theta}{1 + \frac{1}{2}} = \frac{2}{3} \frac{g \sin \theta}{g \sin \theta} = \frac{2}{3} \quad (= 0.667 g \sin \theta) \quad \text{... (3)}
\]

(iii) Spherical shell (hollow sphere) : \( I = \frac{2}{3} MR^2 \), so that \( \beta = \frac{2}{3} \).
\[
\therefore \quad \alpha = \frac{g \sin \theta}{1 + \frac{2}{3}} = \frac{3}{5} \frac{g \sin \theta}{g \sin \theta} = \frac{3}{5} \quad (= 0.6 g \sin \theta) \quad \text{... (4)}
\]

(iv) Solid sphere : \( I = \frac{2}{5} MR^2 \), so that \( \beta = \frac{2}{5} \).
\[
\therefore \quad \alpha = \frac{g \sin \theta}{1 + \frac{2}{5}} = \frac{5}{7} \frac{g \sin \theta}{g \sin \theta} = \frac{5}{7} \quad (= 0.714 g \sin \theta) \quad \text{... (5)}
\]

Q. 144. A spherical shell rolls down a plane inclined at 30° to the horizontal. What is its acceleration? (1 mark)

Ans. The acceleration of the spherical shell,
\[
a = \frac{3}{5} g \sin \theta = 0.6 g \sin 30^\circ = 0.6 g \times \frac{1}{2} = 0.3 g
\]

Q. 145. A spherical shell and a uniform solid sphere roll down the same inclined plane. Compare their accelerations. (1 mark)

Ans. The ratio of the accelerations, in the usual notation,
\[
\frac{a_{\text{shell}}}{a_{\text{sphere}}} = \frac{\frac{3}{5} g \sin \theta}{\frac{2}{5} g \sin \theta} = \frac{21}{25} = 0.84
\]

Q. 146. A ring, a solid cylinder and a solid sphere have exactly the same mass \( M \) and the same radius \( R \). They are simultaneously released from rest at the top of an inclined plane. If they roll down without slipping, would they arrive at the bottom at the same time? Justify your answer. (3 marks)

Ans. Refer to the answer to Q. 143 and continue:
Equations (2), (3) and (4) show that the sphere accelerates the fastest and therefore reaches the bottom first before the solid cylinder, while the ring’s acceleration is the least and therefore reaches last.
Q. 147. A solid sphere, starting from rest, rolls down two different inclined planes from the same height but with different angles of inclination \( \theta_1 > \theta_2 \). On which plane will the sphere take longer time to roll down? (2 marks) 

Ans. Let \( L_1 \) and \( L_2 \) be the distances rolled down by the sphere along the corresponding inclines from the same height \( h \).

\[ \therefore L_1 \sin \theta_1 = L_2 \sin \theta_2 = h \]

Fig. 1.62

Since \( \theta_2 < \theta_1 \), \( \frac{L_1}{L_2} \frac{\sin \theta_2}{\sin \theta_1} < 1 \)

\[ v = \sqrt{\frac{2gh}{1 + (\frac{k^2}{R^2})}} \]

Therefore, \( h \) being the same, \( v_{CM1} = v_{CM2} \)

Since \( a_{CM} = \frac{g \sin \theta}{1 + (\frac{k^2}{R^2})} \), \( a_{CM1} > a_{CM2} \)

\[ v = u + at \]

With \( u = 0 \) and \( v_{CM1} = v_{CM2} \), \( a_{CM1}t_1 = a_{CM2}t_2 \).

\[ \therefore \frac{t_1}{t_2} = \frac{a_{CM2}}{a_{CM1}} < 1 \]

\[ \therefore t_1 < t_2 \]

The sphere will take longer time to roll down from the same height on the plane with smaller inclination.

Q. 148. Two circular discs A and B, having the same mass, have four identical small circular discs placed on them, as shown in the diagram. They are simultaneously released from rest at the top of an inclined plane. If the discs roll down without slipping, which disc will reach the bottom first? (3 marks)

Ans. The disc A has the smaller discs closer to the centre than disc B. Hence, the moment of inertia of disc A \( (I_A) \) is less than that of disc B \( (I_B) \). Suppose the larger discs have radius \( R \), the smaller discs have mass \( m \) and radius \( r \), and the centre of each smaller disc on disc A is at a distance \( x \) from the centre. Then, \( x = \sqrt{2r} \) and, it can be shown that, \( I_B - I_A = 4m[R^2 - (x - r)^2] > 0 \).

Each composite disc is equivalent to a disc of the same radius \( R \) and mass \( M' = M + 4m \), where \( m \) is the mass of each smaller disc, but of different thicknesses.

Suppose, starting from rest, the composite discs roll down the same distance \( L \) along a plane inclined at an angle \( \theta \), their respective accelerations will be

\[ a_A = \frac{g \sin \theta}{1 + (\frac{I_A}{MR^2})} \]
\[ a_B = \frac{g \sin \theta}{1 + (\frac{I_B}{MR^2})} \]

so that, the respective times taken to travel the distance \( L \) are

\[ t_A = \sqrt{\frac{2L}{a_A}} = \sqrt{\frac{2L}{\frac{g \sin \theta}{1 + (\frac{I_A}{MR^2})}}} \]
\[ t_B = \sqrt{\frac{2L}{a_B}} = \sqrt{\frac{2L}{\frac{g \sin \theta}{1 + (\frac{I_B}{MR^2})}}} \]

\[ \therefore t_A < t_B \]

i.e., the disc A will reach the bottom first.

Solved Problems 1.11 – 1.11.1

Q. 149. Solve the following:

(1) A lawn roller of mass 80 kg, radius 0.3 m and moment of inertia 3.6 kg·m\(^2\), is drawn along a level surface at a constant speed of 1.8 m/s. Find (i) the translational kinetic energy (ii) the rotational kinetic energy (iii) the total kinetic energy of the roller. (3 marks)
Solution:

Data: $M = 80 \text{ kg}$, $R = 0.3 \text{ m}$, $I = 3.6 \text{ kg·m}^2$, $v = 1.8 \text{ m/s}$

(i) The translational kinetic energy of the centre of mass of the roller,

$$E_{\text{tran}} = \frac{1}{2} Mv^2 = \frac{1}{2} \times 80 \times (1.8)^2 = 129.6 \text{ J}$$

(ii) The rotational kinetic energy about the roller’s axle,

$$E_{\text{rot}} = \frac{1}{2} I\omega^2 = \frac{1}{2} \left( \frac{v}{R} \right)^2 = \frac{1}{2} \times 3.6 \times \left( \frac{1.8}{0.3} \right)^2$$

$$= 1.8 \times 36 = 64.8 \text{ J}$$

(iii) The total kinetic energy of the roller,

$$E = E_{\text{tran}} + E_{\text{rot}} = 129.6 + 64.8 = 194.4 \text{ J}$$

(2) A solid sphere of mass 1 kg rolls on a table with linear speed 2 m/s, find its total kinetic energy.

Solution:

Data: $M = 1 \text{ kg}$, $v = 2 \text{ m/s}$

The total kinetic energy of a rolling body,

$$E = \frac{1}{2} Mv^2 \left( 1 + \frac{k^2}{R^2} \right)$$

For a solid sphere, $k^2 = \frac{2}{5} R^2$

$$\therefore E = \frac{1}{2} Mv^2 \left( 1 + \frac{2}{5} \right) = \frac{7}{10} \times 1 \times 2^2 = \frac{7 \times 4}{10} = 2.8 \text{ J}$$

(3) A ring and a disc having the same mass roll on a horizontal surface without slipping with the same linear velocity. If the total KE of the ring is 8 J, what is the total KE of the disc? (2 marks)

Solution:

Data: $M_{\text{ring}} = M_{\text{disc}} = M$, $v_{\text{ring}} = v_{\text{disc}} = v$, $E_{\text{ring}} = 8 \text{ J}$

The total kinetic energies of rolling without slipping on a horizontal surface,

$$E_{\text{ring}} = Mv^2 \text{ and } E_{\text{disc}} = \frac{3}{4} \times Mv^2$$

since they have the same mass and linear velocity.

$$\therefore E_{\text{disc}} = \frac{3}{4} E_{\text{ring}} = \frac{3}{4} \times 8 = 6 \text{ J}$$

(4) A solid cylinder, of mass 2 kg and radius 0.1 m, rolls down an inclined plane of height 3 m. Calculate its rotational energy when it reaches the foot of the plane. (2 marks)

Solution:

Data: $M = 2 \text{ kg}$, $R = 0.1 \text{ m}$, $h = 3 \text{ m}$, $g = 10 \text{ m/s}^2$

The MI of a cylinder, $I = \frac{1}{2} MR^2$

$$\therefore \beta = \frac{I}{MR^2} = \frac{1}{2}$$

At the foot of an inclined plane, the speed of a circular body rolling down the inclined plane from a height $h$ is

$$v = \sqrt{\frac{2gh}{1 + \beta}}$$

The rotational energy of the sphere,

$$E = \frac{1}{2} I\omega^2 = \frac{1}{2} \left( \frac{1}{2} MR^2 \right) \frac{v^2}{R^2} = \frac{1}{4} Mv^2$$

$$= \frac{M}{4} \times \frac{2gh}{1 + \frac{1}{2}} = \frac{M}{4} \times \frac{2gh}{1 + \frac{1}{2}}$$

$$= \frac{M}{3} \times \frac{2}{3} (10)(3) = 20 \text{ J}$$

The rotational energy of the sphere,

$$E_{\text{rot}} = \frac{1}{2} I\omega^2 = \frac{1}{2} \left( \frac{2}{5} MR^2 \right) \frac{v^2}{R^2}$$

$$= \frac{M}{5} \left( \frac{2gh}{1 + \frac{1}{2}} \right) = \frac{M}{5} \left( \frac{2gh}{1 + \frac{1}{2}} \right)$$

$$= \frac{M}{5} \left( \frac{2gh}{1 + \frac{1}{2}} \right)$$

$$= \frac{2(2)(9.8)(3)}{7} = 16.8 \text{ J}$$

* (5) Starting from rest, a body rolls down along an incline that rises by 3 in every 5 along the plane. The body gains a speed of $\sqrt{10} \text{ m/s}$ as it travels a distance of $\frac{5}{3} \text{ m}$ along the incline. What can be the possible shape(s) of the body? (2 marks)

Solution:

Data: $\theta = \frac{3}{5}$, $u = 0$, $v = \sqrt{10} \text{ m/s}$, $L = \frac{5}{3} \text{ m}$, $g = 10 \text{ m/s}^2$

$$v = \sqrt{\frac{2gL \sin \theta}{1 + \left( \frac{k^2}{R^2} \right)}} = \sqrt{\frac{2gL \sin \theta}{1 + \beta}}$$

1. ROTATIONAL DYNAMICS
Therefore, the body rolling down is either a ring or a cylindrical shell.

(6) A solid sphere rolls up a plane inclined at 45° to the horizontal. If the speed of its centre of mass at the bottom of the plane is 5 m/s, find how far the sphere travels up the plane. (2 marks)

Solution:

Data: \( v = 5 \text{ m/s}, \theta = 45°, g = 9.8 \text{ m/s}^2 \)

The total energy of the sphere at the bottom of the plane is

\[
E = \frac{7}{10} M v^2
\]

where \( M \) is the mass of the sphere.

In rolling up the incline through a vertical height \( h \), it travels a distance \( L \) along the plane. Then,

\[
h = L \sin 45° = \frac{L}{\sqrt{2}}
\]

By conservation of energy,

\[
Mg h = E
\]

\[
\therefore M g \frac{L}{\sqrt{2}} = \frac{7}{10} M v^2
\]

\[
\therefore L = \frac{7 \sqrt{2}}{10} \frac{v^2}{g} = \frac{7 \times 1.414 \times (5)^2}{9.8} = \frac{4.949 \times 5}{9.8} = 2.526 \text{ m}
\]

The sphere travels 2.526 m up the plane.

**Multiple Choice Questions**

Q. 150. Choose the correct option: (1 mark each)

Note: Students are expected to write the option number [viz., (A), (B), (C) or (D)] as well as the option in full.

1. When seen from below, the blades of a ceiling fan are seen to be revolving anticlockwise and their speed is decreasing. Select the correct statement about the directions of its angular velocity and angular acceleration.

   (A) Angular velocity upwards, angular acceleration downwards.
   (B) Angular velocity downwards, angular acceleration upwards.
   (C) Both angular velocity and angular acceleration upwards.
   (D) Both angular velocity and angular acceleration downwards.

2. A particle of mass 1 kg is tied to a string 1.2 m long and whirled in vertical circular motion under gravity. The minimum speed of the particle is 5 m/s. Consider the following statements:

   (I) The maximum speed must be \( 5 \sqrt{5} \text{ m/s} \).
   (II) The difference between maximum and minimum tensions in the string is 60 N.

Select the correct option:

   (A) Only the statement (I) is correct.
   (B) Only the statement (II) is correct.
   (C) Both the statements are correct.
   (D) Both the statements are incorrect.

3. Select the correct statement about MI of a body in terms of its mass \( M \) and appropriate dimensions (such as \( R, L \)):

   (A) Different objects must have different expressions for their MI.
   (B) When rotating about their respective central axis, a hollow right circular cone and a disc have the same expression for the MI.
   (C) The expression for the MI of a parallelepiped rotating about the transverse axis passing through its centre includes its depth.
   (D) The expression for the MI of a rod and that of a plane sheet is the same about a transverse axis.

4. In a certain unit, the radius of gyration of a uniform disc about its central and transverse axis is \( \sqrt{2.5} \) units. Its radius of gyration about a tangent in its plane (in the same unit) must be

   (A) \( \sqrt{5} \) (B) 2.5 (C) 2\sqrt{2.5} (D) \( \sqrt{12.5} \).

5. Consider the following cases:

   (i) A planet revolving in an elliptical orbit.
   (ii) A planet revolving in a circular orbit.
A thin-walled hollow cylinder is rolling down an incline without slipping. At any instant, the ratio of KE_{rotational} : KE_{translational} : KE_{total} is
(A) 1 : 1 : 2 (B) 1 : 2 : 3 (C) 1 : 1 : 1 (D) 2 : 1 : 3.

7. The bulging of the Earth at the equator and flattening at the poles is due to
(A) centripetal force (B) centrifugal force (C) gravitational force (D) electrostatic force.

8. A body of mass 0.4 kg is revolved in a horizontal circle of radius 5 m. If it performs 120 rpm, the centripetal force acting on it is
(A) 4\pi^2 N (B) 8\pi^2 N (C) 16\pi^2 N (D) 32\pi^2 N.

9. Two particles with their masses in the ratio 2 : 3 perform uniform circular motion with orbital radii in the ratio 3 : 2. If the centripetal force acting on them is the same, the ratio of their speeds is
(A) 4 : 9 (B) 1 : 1 (C) 3 : 2 (D) 9 : 4.

10. When a motorcyclist takes a circular turn on a level race track, the centripetal force is
(A) the resultant of the normal reaction and frictional force (B) the horizontal component of the normal reaction (C) the frictional force between the tyres and road (D) the vertical component of the normal reaction.

11. The maximum speed with which a car can be driven safely along a curved road of radius 17.32 m and banked at 30° with the horizontal is \[ g = 10 \text{ m/s}^2 \]
(A) 5 m/s (B) 10 m/s (C) 15 m/s (D) 20 m/s.

12. A track for a certain motor sport event is in the form of a circle and banked at an angle \( \theta \). For a car driven in a circle of radius \( r \) along the track at the optimum speed, the periodic time is
(A) \( \sqrt{\frac{r}{g}} \) (B) \( 2\pi \sqrt{\frac{r}{g}} \) (C) \( 2\pi \sqrt{\frac{r}{g \tan \theta}} \) (D) \( 2\pi \sqrt{\frac{r \tan \theta}{g}} \).

13. The period of a conical pendulum in terms of its length (\( l \)), semivertical angle (\( \theta \)) and acceleration due to gravity (\( g \)), is
(A) \( \frac{1}{2\pi} \sqrt{\frac{l \cos \theta}{g}} \) (B) \( \frac{1}{2\pi} \sqrt{\frac{l \sin \theta}{g}} \) (C) \( 4\pi \sqrt{\frac{l \cos \theta}{4g}} \) (D) \( 4\pi \sqrt{\frac{l \tan \theta}{g}} \).

14. A conical pendulum of string length \( L \) and bob of mass \( m \) performs UCM along a circular path of radius \( r \). The tension in the string is
(A) \( \frac{mgL}{\sqrt{L^2 - r^2}} \) (B) \( \frac{mgL}{\sqrt{L^2 + r^2}} \) (C) \( \frac{mgL}{2r} \) (D) \( \frac{mg \tan \theta}{L} \).

15. The centripetal acceleration of the bob of a conical pendulum is
(A) \( \frac{rg}{\cos \theta} \) (B) \( \frac{rg}{L} \) (C) \( \frac{g}{L} \) (D) \( \frac{rg}{L \cos \theta} \).

16. A small object tied at the end of a string is to be whirled in a vertical circle of radius \( r \). If its speed at the lowest point is \( 2\sqrt{gr} \), then
(A) the string will be slack at the lowest point (B) it will not reach the midway point (C) its speed at the highest point will be \( \sqrt{gr} \) (D) it will just reach the highest point with zero speed.

17. A small bob of mass \( m \) is tied to a string and revolved in a vertical circle of radius \( r \). If its speed at the highest point is \( \sqrt{3gr} \), the tension in the string at the lowest point is
(A) 5mg (B) 6mg (C) 7mg (D) 8mg.

18. A small object, tied at the end of a string of length \( r \), is launched into a vertical circle with a speed \( 2\sqrt{gr} \) at the lowest point. Its speed when the string is horizontal is
(A) \( > \sqrt{3gr} \) (B) \( = \sqrt{3gr} \) (C) \( = \sqrt{2gr} \) (D) \( 0 \).

19. Two bodies with moments of inertia \( I_1 \) and \( I_2 \) \((l_1 > l_2)\) rotate with the same angular momentum. If \( E_1 \) and \( E_2 \) are their rotational kinetic energies, then
(A) \( E_2 > E_1 \) (B) \( E_2 = E_1 \) (C) \( E_2 < E_1 \) (D) \( E_2 \leq E_1 \).

1. ROTATIONAL DYNAMICS 79
20. The radius of gyration \( k \) for a rigid body about a given rotation axis is given by

\[
(A) \ k = \frac{1}{M} \int r \, dm \quad (B) \ k^2 = \frac{1}{M} \int r^2 \, dm \\
(C) \ k^2 = \frac{1}{M} \int r \, dm \quad (D) \ k = \frac{1}{M} \int r^2 \, dm.
\]

21. Three point masses \( m, 2m \) and \( 3m \) are located at the three vertices of an equilateral triangle of side \( l \). The moment of inertia of the system of particles about an axis perpendicular to their plane and equidistant from the vertices is

\[
(A) \ 2ml^2 \quad (B) \ 3ml^2 \quad (C) \ 2\sqrt{3}ml^2 \quad (D) \ 6ml^2.
\]

22. The moment of inertia of a thin uniform rod of mass \( M \) and length \( L \), about an axis passing through a point midway between the centre and one end, and perpendicular to its length, is

\[
(A) \ \frac{48}{7} ML^2 \quad (B) \ \frac{7}{48} ML^2 \quad (C) \ \frac{1}{48} ML^2 \quad (D) \ \frac{1}{16} ML^2.
\]

23. A thin uniform rod of mass \( M \) and length \( L \) has a small block of mass \( m \) attached at one end. The moment of inertia of the system about an axis through its CM and perpendicular to the length of the rod is

\[
(A) \ \frac{13}{12} ML^2 \quad (B) \ \frac{1}{3} ML^2 \quad (C) \ \frac{5}{24} ML^2 \quad (D) \ \frac{7}{48} ML^2.
\]

24. A thin wire of length \( L \) and uniform linear mass density \( \lambda \) is bent into a circular ring. The MI of the ring about a tangential axis in its plane is

\[
(A) \ \frac{3\lambda L^2}{8\pi^2} \quad (B) \ \frac{8\pi^2}{3L^3} \quad (C) \ \frac{3\lambda L^3}{8\pi^2} \quad (D) \ \frac{8\pi^2}{3L^2}.
\]

25. When a planet in its orbit changes its distance from the Sun, which of the following remains constant? 

(A) The moment of inertia of the planet about the Sun 
(B) The gravitational force exerted by the Sun on the planet 
(C) The planet’s speed 
(D) The planet’s angular momentum about the Sun

26. If \( L \) is the angular momentum and \( I \) is the moment of inertia of a rotating body, then \( \frac{L^2}{2I} \) represents its

(A) rotational PE 
(B) total energy 
(C) rotational KE 
(D) translational KE.

27. A thin uniform rod of mass 3 kg and length 2 m rotates about an axis through its CM and perpendicular to its length. An external torque changes its frequency by 15 Hz in 10s. The magnitude of the torque is

\[
(A) \ 3.14 \text{ N·m} \quad (B) \ 6.28 \text{ N·m} \\
(C) \ 9.42 \text{ N·m} \quad (D) \ 12.56 \text{ N·m}.
\]

28. The flywheel of a motor has mass 300 kg and radius of gyration 1.5 m. The motor develops a constant torque of 2000 N·m and the flywheel starts from rest. The work done by the motor during the first 4 revolutions is

\[
(A) \ 2 \text{ kJ} \quad (B) \ 8 \text{ kJ} \quad (C) \ 8\pi \text{ kJ} \quad (D) \ 16\pi \text{ kJ}.
\]

29. Two uniform solid spheres, of the same mass but radii in the ratio \( R_1 : R_2 = 1 : 2 \), roll without slipping on a plane surface with the same total kinetic energy. The ratio \( \omega_1 : \omega_2 \) of their angular speed is

\[
(A) \ 2 : 1 \quad (B) \ \sqrt{2} : 1 \quad (C) \ 1 : 1 \quad (D) \ 1 : 2.
\]

30. A circularly symmetric body of radius \( R \) and radius of gyration \( k \) rolls without slipping along a flat surface. Then, the fraction of its total energy associated with rotation is \( [c = k^2 / R^2] \)

\[
(A) \ c \quad (B) \ \frac{c}{1+c} \quad (C) \ \frac{1}{c} \quad (D) \ \frac{1}{1+c}.
\]

**Answers**

1. (A) Angular velocity upwards, angular acceleration downwards.  2. (C) Both the statements are correct.  
3. (B) When rotating about their respective central axis, a hollow right circular cone and a disc have the same expression for the MI.  
4. (B) 2.5  5. (C) For both (i) and (ii).  6. (D) 2 : 1 : 3  7. (B) Centrifugal force  
8. (D) 32\pi N  9. (C) 3 : 2  10. (C) The frictional force between the tyres and road  
11. (B) 10 m/s  
12. (C) 2\pi \sqrt{\frac{r}{g \tan \theta}}  13. (C) 4\pi \sqrt{\frac{1}{4g} \cos \theta}  
14. (A) \frac{mgL}{\sqrt{L^2 - r^2}}  15. (D) \frac{rg}{L \cos \theta}  
16. (D) It will just reach the highest point with zero speed.  17. (D) 8mg  
18. (C) \sqrt{2gr}  19. (A) \frac{E_2}{E_1}  20. (B) \frac{k^2}{M} = \int r^2 \, dm  
21. (A) 2ml^2  22. (B) \frac{7}{48} ML^2  23. (C) \frac{5}{24} ML^2  24. (C) \frac{3\lambda L^3}{8\pi^2}  
25. (D) The planet’s angular momentum about the Sun  26. (C) Rotational KE  27. (C) 9.42 N·m  
28. (D) 16\pi \text{ kJ}  29. (A) 2 : 1  30. (B) \frac{c}{1+c}.
1. An object of mass 200 grams is tied to the end of a string and revolved in a horizontal circle of radius 1.5 m. If it performs 120 revolutions per minute, calculate (i) its angular speed (ii) its linear speed (iii) the centripetal force. (3 marks)  
(Ans. 12.57 rad/s, 18.85 m/s, 47.4 N)

2. A particle performs circular motion of radius 2 m with a constant angular acceleration of 0.4 rad/s². If its initial angular speed is 0.4 rad/s, find its (i) tangential acceleration (ii) angular speed after 2 s (iii) angular displacement in 2 s. (3 marks)  
(Ans. 0.8 m/s², 1.2 rad/s, 1.6 rad)

3. The angular speed of a particle performing circular motion at a given instant is 4 rad/s. If it has a constant angular acceleration of 5 rad/s² and the radius of the path is 20 cm, find its (i) angular speed after 0.5 s (ii) tangential acceleration. (3 marks)  
(Ans. 6.5 rad/s, 1 m/s²)

4. The blades of a windmill have a radius of 40 m. At top speed, a blade tip has a speed of 360 m/s. The speed reduces to 280 m/s in 10 s as the windmill slows to rest at constant acceleration. (i) How much further time elapses before the blades come to rest? (ii) How many revolutions do the blades make in coming to rest from top speed? (3 marks)  
(Ans. 35 s, 1273)

5. A fan is rotating at 90 rpm. When switched off, it stops after 21 revolutions. Assuming a constant frictional torque, calculate the time taken by it to come to a stop. (2 marks) (Ans. 28 s)

6. A motorcycle and its rider, together weighing 250 kg, move along a horizontal curve of radius 200 m with a constant speed of 36 km/h. Calculate (i) its angular speed (ii) its centripetal acceleration (iii) the centripetal force. (3 marks)  
(Ans. 0.05 rad/s, 0.5 m/s², 125 N)

7. A coin is kept on a gramophone disc with its centre at a distance of 5 cm from the centre of the disc and the disc is set into rotation. The coin begins to slip when the disc just reaches a speed of 45 rotations per minute. Find the coefficient of friction between the coin and the disc. (2 marks) (Ans. 0.1133)

8. Find the maximum speed with which a car can be safely driven along a curve of radius 30 m, if the coefficient of friction between the wheels and the road is 0.3. (2 marks) (Ans. 9.39 m/s)

9. A 70 kg man stands against the inner wall of a hollow cylindrical drum (the Rotor) of radius 3 m rotating about its vertical axis. The coefficient of friction between the wall and his clothing is 0.15. What is the minimum rotational speed of the cylinder to enable the man to remain stuck to the wall without falling when the floor is suddenly removed? (2 marks) (Ans. 4.714 rad/s or 45 rpm)

10. A motorcyclist (treated as a particle) is to ride in horizontal circles inside the cylindrical wall of a well-of-death of radius 4 m. The coefficient of static friction between the tyres and the wall is 0.4. Calculate the minimum speed and frequency necessary to perform the stunt. (2 marks) (Ans. 10 m/s, ~0.4 rps)

11. Calculate the angle of banking of a smooth curved road of radius 100 m, if vehicles can safely travel along it with a speed of 108 km/h. (2 marks) (Ans. 42°34’)

12. The road course of a Formula One race track has a turn of radius 400 m banked at an angle of 9°. Under dry weather conditions, the friction between the tyres and the road surface is 0.9. Calculate the (i) optimum speed (ii) maximum safe speed at the turn. (3 marks) (Ans. 24.92 m/s, 69.55 m/s)

13. A circular race track of radius 300 m is banked at an angle of 15°. If the coefficient of friction between the tyres of a race car and the road surface is 0.2, what is (a) the optimum speed of the car to avoid wear and tear on its tyres (b) the maximum permissible speed to avoid slipping? (3 marks)  
(Ans. 28.07 m/s, 38.12 m/s)

14. A motorcyclist drives along a circular track of radius 50 m with a speed of 54 km/h. Through what angle should he lean inwards to keep his balance? (2 marks) (Ans. 24°40’
15. A motorcyclist going round a circular path has to lean inwards making an angle of 21°49' with the vertical in order to maintain his balance. Find the speed of the motorcyclist, if the circular path is 1 km long. (2 marks) (Ans. 25 m/s)

16. Find the angle of banking of a railway track of radius of curvature 1500 m, if the optimum speed of the train is 24 m/s. Also find the elevation of the outer rail over the inner rail, if the two rails are 1.6 m apart. (3 marks) (Ans. 2°15'; 0.0627 m)

17. A train of mass 10^3 kg rounds a curve of radius 150 m at a speed of 20 m/s. Find the horizontal thrust on the outer rail, if the track is not banked. At what angle must the track be banked in order that there is no thrust on the rail? (Hint: The horizontal thrust is equal to the magnitude of the centripetal force.) (3 marks) (Ans. 2.667 x 10^5 N; 15°13')

18. A racing track of radius of curvature 9.9 m is banked at tan^{-1} 0.5. The coefficient of static friction between the track and the tyres of a vehicle is 0.2. Determine the speed limits with 10% margin. (3 marks each) (Ans. 5.716 m/s; 7.896 m/s)

19. The figure shows vertical section of a merry-go-round in which the ‘initially vertical’ rods are inclined with the vertical at 37° during rotation. Calculate the frequency of rotation of the merry-go-round. [Take g = π^2 m/s^2 and sin 37° = 0.6] (3 marks) (Ans. 0.25 rps)

20. A conical pendulum has length 100 cm and the angle made by the string with the vertical is 10°. The mass of the bob is 200 grams. Find (i) the centripetal force on the bob (ii) the frequency of circular motion of the bob. (3 marks) (Ans. 0.3455 N, 0.5024 Hz)

21. Semivertical angle of the conical section of a funnel is 37°. There is a small ball kept inside the funnel. On rotating the funnel, the maximum speed that the ball can have in order to remain in the funnel is 2 m/s. Calculate inner radius of the brim of the funnel. Is there any limit upon the frequency of rotation? How much is it? Is it lower or upper limit? Give a logical reasoning. [Take g = 10 m/s^2, sin 37° = 0.6] (3 marks) (Ans. ∼1 m/s)

22. A motorcyclist rides in vertical circles in the Globe of Death—a steel spherical cage 6 m wide. The motorcyclist and the bike have a total mass of 180 kg. Find the normal reaction exerted by the cage midway up if his speed there is 54 kmph. (2 marks) (Ans. 1.35 kN)

23. A pilot of mass 80 kg in a fighter aircraft executes a vertical circle of radius 2.5 km at a constant speed of 250 m/s. Calculate the force exerted by the seat on the pilot (i) at the bottom of the loop (ii) at the top of the loop. [g = 10 m/s^2] (3 marks) (Ans. 2.8 kN, 1.2 kN)

24. A small stone of mass 20 g is tied to a practically massless inextensible string and whirled in vertical circles. (i) The speed of the stone is 8 m/s when the centripetal force is exactly equal to the force due to the tension. Calculate minimum and maximum kinetic energies of the stone during the entire circle. (ii) Taking the angular position of the string, θ, equal to zero when the stone is at the lowermost position, determine θ when the force due to tension is numerically equal to weight of the stone. [Take g = 10 m/s^2 and the length of the string = 1.8 m] (3 marks) (Ans. 0.28 J, 1 J, 148°25')

25. A light metre rod has two point masses, each of 2 kg, fixed at its ends. If the system rotates about its centre of mass with an angular speed of 0.5 rad/s, find its rotational KE. (2 marks) (Ans. 0.125 J)

26. (a) Four identical masses m are fixed at the corners of sides b. Calculate the MI of this system for rotation about axes AA', BB' and CCI (2 marks)
(b) If m = 1.2 kg and b = 20 cm, determine the MI of this system for rotation about (i) an axis along the diagonal of this square (ii) an axis parallel to a side and passing through the centre of the square (iii) an axis through the centre of the
square and perpendicular to its plane. (2 marks)

[Ans. (a) 2mb², mb², 2mb²; (b) 0.048 kg·m², 0.048 kg·m², 0.096 kg·m²]

27. Two masses $M_1$ and $M_2$ are attached to a massless rod of length $L$. (a) Find the MI of the system about an axis perpendicular to the rod and through the mass $M_2$. (b) Find an expression for the moment of inertia of the system for rotation about the CM, about an axis perpendicular to the rod. (3 marks)

[Ans. $M_1 L^2$, $(M_1 + M_2) \frac{L^2}{4}$]

28. Three particles each of mass $m$ are attached to a thin rod of mass $3m$ as shown. Find the moment of inertia of the system for rotation about the end O. (2 marks) (Ans. 23 mL²)

29. A flywheel has mass 2 kg and radius of gyration 0.2 m. Calculate its kinetic energy of rotation when it makes 5 rps. (2 marks) (Ans. 39.44 J)

30. Calculate the moment of inertia of a hoop of mass 0.4 kg and radius 0.3 m about (i) its diameter (ii) a tangent in its plane. (3 marks) (Ans. 0.018 kg·m², 0.054 kg·m²)

31. A uniform disc of mass 5 kg has a radius of 0.5 m. Find its MI about an axis through a point on its circumference and perpendicular to its plane. (2 marks) (Ans. 1.875 kg·m²)

32. The radius of gyration of a body about an axis at a distance of 12 cm from its centre of mass is 13 cm. Find its radius of gyration about a parallel axis through its centre of mass. (2 marks) (Ans. 5 cm)

33. The moment of inertia of a disc about an axis passing through its centre and perpendicular to its plane is 20 kg·m². Determine its moment of inertia about an axis (1) coinciding with a tangent perpendicular to its plane (2) perpendicular to its plane and through a point midway between the centre and a point on the circumference. (3 marks)

(Ans. 60 kg·m², 30 kg·m²)

34. Find the radius of gyration of a thin uniform rod of length 1 m about an axis perpendicular to its length and through (i) an end (ii) the centre. (3 marks)

(Ans. 0.5774 m, 0.2887 m)

35. A solid cylinder of uniform density has mass $M$, radius $R$ and length $L$ with $R = L/\sqrt{3}$. It rotates about a transverse axis through its centre. (1) Find its radius of gyration in terms of $R$. (2) If $M = 1.5$ kg and $L = 6$ cm, find its MI about the given axis. (3 marks)

(Ans. $R/\sqrt{2}$; $9 \times 10^{-4}$ kg·m²)

36. The flywheel—a heavy disc designed to store rotational energy—in a particular machine is in the form of a uniform 20 kg disc of diameter 50 cm, able to rotate about its own axis. Calculate (i) its kinetic energy when rotating at 1200 rpm (ii) its moment of inertia about a tangent in its plane. [Use $\pi^2 \approx 10$] (3 marks) (Ans. 5000 J)

37. A solid cylinder, of mass 20 kg and radius 25 cm, rotates about its cylinder axis with an angular speed of 100 rad/s. Find (i) its rotational kinetic energy (ii) the magnitude of its angular momentum about that axis. (3 marks) (Ans. 3125 J, 62.5 kg·m·s)

38. Torques of equal magnitude are applied to a hollow cylinder and a solid sphere, both having the same mass and radius. The cylinder is free to rotate about its cylinder axis and the sphere is free to rotate about an axis through its centre. Which of the two will acquire a greater angular speed after a given time? (2 marks) [Hint: $\tau = I_{cyl} \alpha_{cyl} = I_{sph} \alpha_{sph}$] (Ans. Sphere)

39. A flywheel of mass 2 kg and radius 10 cm is subjected to a torque of magnitude 0.1 N·m. Calculate the angular acceleration produced. What is the radius of gyration of the flywheel? (3 marks)

(Ans. 10 rad/s², 0.0707 m)
40. The angular speed of a flywheel changes from 10 rad/s to 20 rad/s in 5 seconds when a torque of 20 N·m is applied to it. Calculate the MI of the flywheel. **(2 marks)** (Ans. 10 kg·m²)

41. A torque of magnitude 2000 N·m on a body produces an angular acceleration of 2 rad/s². Calculate the moment of inertia of the body. **(2 marks)** (Ans. 1000 kg·m²)

42. Calculate the torque necessary to produce an angular acceleration of 25 rad/s² in a flywheel of mass 50 kg and radius of gyration 50 cm about its axis. **(2 marks)** (Ans. 312.5 N·m)

43. A uniform disc of mass 30 kg and radius 0.5 m is belt-driven by a motor. If the disc, starting from rest and accelerating uniformly, attains an angular speed of 20 rot/s in 15 s, what is the tension in the belt? **(2 marks)** (Ans. 62.8 N)

44. Assuming that the effective frictional braking torque on the rotor of a centrifuge is 0.6 N·m and that the rotor has a mass of 1.2 kg and a MI of 4 × 10⁻³ kg·m², determine the time required for the rotor to come to rest if it is spinning at 6000 rpm. **(2 marks)** (Ans. 4.187 s)

45. A cord of negligible mass is wound round a hollow cylinder of mass 3 kg, radius 40 cm and free to rotate about the cylinder axis. (i) What is the angular acceleration of the cylinder if the rope is pulled with a force of 30 N? (ii) What is the linear acceleration of the rope? **(3 marks)** (Ans. 25 rad²/s², 10 m/s²)

46. A flywheel, of mass 20 kg and radius 20 cm and mounted on a horizontal axle with frictionless bearings is initially at rest. A cord of negligible mass is wound round its rim. If the cord is pulled down with a steady force of 25 N, calculate (i) the angular acceleration of the flywheel (ii) its kinetic energy after 2 m of the cord is unwound. **(3 marks)** (Ans. 12.5 rad²/s², 50 J)

47. A 40 W electric motor keeps a flywheel of moment of inertia 10 kg·m² rotating at 20 rad/s. When the motor is switched off, find (i) the work done against friction (ii) the number of rotations completed by the flywheel before coming to rest (iii) the corresponding time. **(3 marks)** (Ans. -2 kJ, 159.1, 100 s)

48. To maintain a rotor at a uniform angular speed of 200 rad/s, a motor needs to transmit a torque of 180 N·m. What is the power required by the motor? Assume that the efficiency of the motor is 100%. [Hint: Uniform angular velocity in the absence of friction implies zero torque. In practice, applied torque is needed to counter frictional torque.] **(2 marks)** (Ans. 36 kW)

49. A ceiling fan having moment of inertia 2 kg·m² attains its maximum frequency of 60 rpm in 2π seconds. Calculate its power rating. **(2 marks)** (Ans. 16π W ≈ 50 W)

50. A child stands at the centre of a frictionless turntable with his two arms outstretched. The turntable is set rotating with an angular speed of 40 rpm. What is the angular speed of the turntable if he folds his hands back, thereby reducing his moment of inertia to 2/5 times the initial value? **(2 marks)** (Ans. 100 rpm)

51. A man stands on a platform rotating at 30 rpm with his arms outstretched and a 5 kg block in each hand. The man then brings his arms close to his body such that the distance of each block from the axis changes from 90 cm to 20 cm. Assume the moment of inertia of the man together with the platform to be constant and equal to 7.6 kg·m². Ignoring friction, what is his new angular speed? **(2 marks)** (Ans. 6.167 rad/s)

52. A wheel is rotating with a frequency of 500 rotations per minute on a shaft. A second identical wheel, initially at rest, is suddenly coupled on the same shaft. What is the frequency of rotation of the resultant combination? Ignore the MI of the shaft. **(2 marks)** (Ans. 250 rpm)

53. The Earth-Sun distance varies from 1.471 × 10⁸ km at perihelion to 1.521 × 10⁸ km at aphelion. The minimum orbital speed of the Earth is 29.3 km/s. Find its maximum orbital speed. **(2 marks)** (Ans. 30.26 km/s)

54. If all of the Earth’s polar ice caps melt due to global warming, it would create a thin spherical shell of water, redistributing the mass at lower latitudes. As a rough estimation, this would increase the moment
of inertia of the Earth by $6.22 \times 10^{32}$ kg·m$^2$. Would the length of the day increase or decrease? By how much?

\[ I_{\text{earth}} = 8.01 \times 10^{37} \text{ kg·m}^2, \quad 1 \text{ day} = 8.64 \times 10^4 \text{ s} \]

(3 marks) (Ans. Increase, 0.671 s)

55. A spherical water balloon is revolving at 60 rpm. In the course of time, 48.8% of its water leaks out. With what frequency will the remaining balloon revolve now? Neglect all non-conservative forces.

(3 marks) (Ans. 3.052 rps)

56. A cylinder of mass 10 kg rolls without slipping on a horizontal surface. At the instant its centre of mass has a speed of 5 m/s, determine (i) the translational kinetic energy (ii) the rotational kinetic energy about the axis through its centre of mass (iii) the total kinetic energy of the cylinder.

(iii) (3 marks)

(Ans. 125 J, 62.5 J, 187.5 J)

57. A uniform disc of mass 1.6 kg and radius 9 cm rolls without slipping across a horizontal floor at a speed of 0.2 m/s. What is its kinetic energy?

(2 marks) (Ans. 48 mJ)

58. A thin ring of mass 100 kg rolls along a horizontal floor such that its CM has a speed of 0.2 m/s. How much work must be done on the ring to stop it?

(2 marks) [Hint: By the work-kinetic energy theorem, \( W = \Delta KE = KE_f - KE_i \)] (Ans. – 4 J)

59. A uniform solid sphere rolls without slipping down a ramp inclined at an angle of $\theta = 30^\circ$. The sphere travels a distance of 2.4 m along the ramp to reach the bottom. Find its speed at the bottom and the acceleration of its centre of mass.

(3 marks) (Ans. 4.1 m/s, 3.5 m/s$^2$)

60. A small solid sphere (mass $m$ and radius $r$) released from rest rolls without slipping along a loop-the-loop track of radius $R$ as shown. Assume $r \ll R$. From what initial height $h$ above the bottom of the track must the sphere be released if it is just able to complete the loop?

(2 marks) (Ans. $h = 2.7 R$)

61. A uniform solid sphere rolls down an inclined plane. Find the angle of inclination for which the centre of mass of the sphere has linear acceleration down the plane of 30% the acceleration due to gravity.

(2 marks) (Ans. 24°50')
1. \( \omega = \frac{d\theta}{dt} = \frac{2\pi}{T} = 2\pi f \quad \therefore \omega = \frac{d\phi}{dt} \)

2. \( v = \omega r \quad a_t = \frac{dv}{dt} = \omega r \)

3. \( T = \frac{2\pi}{\omega} = \frac{2\pi r}{v} \quad \therefore f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{v}{2\pi r} \)

4. \( a_c = \omega v = \frac{\omega^2 r}{r} \quad F_c = ma_c \)

5. Level road :
   \( v_{\text{max}} = \sqrt{r\mu_s g} \)

6. Banked road :
   \( v_{\text{max}} = \sqrt{rg(\mu_s + \tan \theta)} \quad \tan \theta = \frac{v_{\text{opt}}^2}{rg} \)
   \( v_{\text{opt}} = \sqrt{rg \tan \theta} \quad \tan \theta = \frac{v_{\text{opt}}^2}{rg} \)
   \( v_{\text{min}} = \sqrt{rg(\mu_s - \tan \theta)} \quad \tan \theta = \frac{v_{\text{opt}}^2}{rg} \)

7. Conical pendulum :
   \( v = \sqrt{rg \tan \theta} \quad \tan \theta = \frac{v^2}{rg} = \frac{r}{L \cos \theta} \)
   \( \omega = \frac{\sqrt{g}}{\sqrt{L \cos \theta}} \quad T = 2\pi \sqrt{\frac{r}{g \tan \theta}} = 2\pi \sqrt{\frac{L \cos \theta}{g}} \)
   \( F = \frac{mg}{\cos \theta} = mg \sqrt{\left(\frac{r}{L \cos \theta}\right)^2 + 1} = mg (1 + \tan^2 \theta) = mg \sqrt{\frac{2}{1 + \cos 2\theta}} \)

8. Vertical circular motion :
   (i) General :
   
   At the top : \( T_1 \) (or \( N_1 \)) = \( \frac{mv_1^2}{r} = mg \quad \text{PE} = mg \ (2r) \quad \text{KE} = \frac{1}{2}mv_1^2 \)
   
   At the bottom : \( T_2 \) (or \( N_2 \)) = \( \frac{mv_2^2}{r} + mg \quad \text{PE} = 0 \quad \text{KE} = \frac{1}{2}mv_2^2 \)
   
   Midway : \( T_3 \) (or \( N_3 \)) = \( \frac{mv_3^2}{r} \quad \text{PE} = mvr \quad \text{KE} = \frac{1}{2}mv_3^2 \)

   (ii) Critical or minimum values :
   
   At the top : \( T_1 \) (or \( N_1 \)) = 0 \quad v_1 = \sqrt{gr} \quad \text{PE} = mg \ (2r) \quad \text{KE} = \frac{1}{2}mg r \)
   
   At the bottom : \( T_2 \) (or \( N_2 \)) = 6 \ mg \quad v_2 = \sqrt{5gr} \quad \text{PE} = 0 \quad \text{KE} = \frac{5}{2}mg r \)
   
   Midway : \( T_3 \) (or \( N_3 \)) = \( \frac{mv_3^2}{r} = 3 \ mg \quad v_3 = \sqrt{3gr} \quad \text{PE} = mgr \quad \text{KE} = \frac{3}{2}mgr \)

9. Angular kinematical equations (for constant \( \omega \)) :
   \( \omega = \omega_0 + x \omega \)
   \( \theta = \left(\frac{\omega + \omega_0}{2}\right) t = \omega_0 t + \frac{1}{2} x t^2 = \omega t - \frac{1}{2} x t^2 \)
   \( \omega^2 = \omega_0^2 + 2x \theta \)
10. \( I = \sum_{i=1}^{N} m_i r_i^2 \) (system of particles)
\[ I = \int r^2 \, dm \] (rigid body)

11. \( I = Mk^2 \quad k = \sqrt{I/M} \)

12. \( I = I_{CM} + Mh^2 \quad I_z = I_x + I_y \)

13. MI and radius of gyration of some regular bodies of uniform density:

<table>
<thead>
<tr>
<th>Body</th>
<th>Moment of Inertia</th>
<th>Body</th>
<th>Moment of Inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thin rod</td>
<td>( \frac{1}{12} ML^2 )</td>
<td>Thin disc</td>
<td>( \frac{1}{2} MR^2 )</td>
</tr>
<tr>
<td>Transverse through CM</td>
<td></td>
<td>Transverse through CM</td>
<td></td>
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<tr>
<td>Thin ring or hollow cylinder</td>
<td>( \frac{1}{3} ML^2 )</td>
<td>Thin disc</td>
<td>( \frac{3}{2} MR^2 )</td>
</tr>
<tr>
<td>Transverse through an end</td>
<td></td>
<td>Thin disc</td>
<td>( \frac{1}{4} MR^2 )</td>
</tr>
<tr>
<td>Thin ring</td>
<td>( MR^2 )</td>
<td>Thin disc</td>
<td>( \frac{5}{4} MR^2 )</td>
</tr>
<tr>
<td>Transverse through CM</td>
<td></td>
<td>Thin disc</td>
<td></td>
</tr>
<tr>
<td>Thin ring</td>
<td>( \frac{1}{2} MR^2 )</td>
<td>Thin disc</td>
<td>( \frac{1}{2} MR^2 )</td>
</tr>
<tr>
<td>Diameter</td>
<td></td>
<td>Thin disc</td>
<td></td>
</tr>
<tr>
<td>Thin ring</td>
<td>( \frac{3}{2} MR^2 )</td>
<td>Thin disc</td>
<td>( M\left(\frac{R^2}{4} + \frac{L^2}{12}\right) )</td>
</tr>
<tr>
<td>Tangent in its plane</td>
<td></td>
<td>Thin disc</td>
<td></td>
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<tr>
<td>Body</td>
<td>Moment of inertia</td>
<td>Body</td>
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<tr>
<td><strong>Thick walled cylinder</strong></td>
<td>( \frac{1}{2} M \left( R_1^2 + R_2^2 \right) )</td>
<td><strong>Thin rectangular plate</strong></td>
<td>( \frac{1}{12} M \left( l^2 + b^2 \right) )</td>
</tr>
<tr>
<td></td>
<td>( \frac{1}{12} M \left[ 3 \left( R_1^2 + R_2^2 \right) + L^2 \right] )</td>
<td></td>
<td><strong>Roll (Transverse axis through CM)</strong></td>
</tr>
<tr>
<td><strong>Thin spherical shell</strong></td>
<td>( \frac{2}{3} M R^2 )</td>
<td><strong>Hollow cone</strong></td>
<td>( \frac{1}{2} M R^2 )</td>
</tr>
<tr>
<td></td>
<td>( \frac{5}{3} M R^2 )</td>
<td><strong>Solid cone</strong></td>
<td>( \frac{3}{10} M R^2 )</td>
</tr>
<tr>
<td><strong>Solid sphere</strong></td>
<td>( \frac{2}{5} M R^2 )</td>
<td><strong>Rectangular bar</strong></td>
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<tr>
<td></td>
<td>( \frac{7}{5} M R^2 )</td>
<td>( I_{\text{yaw}} = \frac{1}{12} M \left( l^2 + b^2 \right) )</td>
<td></td>
</tr>
<tr>
<td><strong>Thick spherical shell</strong></td>
<td>( \frac{2}{5} M \left( \frac{R_2^5 - R_1^5}{R_2^3 - R_1^3} \right) )</td>
<td>( I_{\text{roll}} = \frac{1}{12} M \left( b^2 + w^2 \right) )</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>( I_{\text{pitch}} = \frac{1}{12} M \left( w^2 + l^2 \right) )</td>
<td></td>
</tr>
</tbody>
</table>
14. \( \vec{\tau} = I_2 \vec{\omega} \)

15. \( E_{\text{rot}} = \frac{1}{2} I_2 \omega_2^2 = 2\pi^2 I_f^2 = 2\pi^2 \frac{I}{I_f^2} = \frac{1}{2} I_\omega \)

16. \( E_{\text{rolling}} = \frac{1}{2} M v^2 + \frac{1}{2} I_2 \omega_2^2 = \frac{1}{2} M v^2 \left( 1 + \frac{k^2}{R^2} \right) = \frac{1}{2} M v^2 \left( 1 + \frac{1}{MR^2} \right) = \frac{1}{2} M v^2 (R^2 + k^2) \)

17. On rolling down an inclined plane, \( v = \sqrt{\frac{2gh}{1 + \beta}} \quad a = \frac{g}{1 + \beta} \sin \theta \quad \text{where} \quad h = L \sin \theta \quad \text{and} \quad \beta = \frac{k^2}{R^2} = \frac{1}{MR^2} \)

\[
\begin{align*}
 v_{\text{ring}} &= \sqrt{gh} \\
 v_{\text{disc}} &= v_{\text{cylinder}} = \frac{2}{\sqrt{3}} gh \\
 v_{\text{sphere}} &= \frac{10}{7} gh \\
 v_{\text{shell}} &= \frac{6}{5} gh \\
 a_{\text{ring}} &= \frac{1}{2} g \sin \theta \\
 a_{\text{disc}} &= a_{\text{cylinder}} = \frac{2}{3} g \sin \theta \\
 a_{\text{sphere}} &= \frac{5}{7} g \sin \theta \\
 a_{\text{shell}} &= \frac{3}{5} g \sin \theta
\end{align*}
\]

18. Work done by a constant external torque, \( W = \tau \theta = \Delta KE_{\text{rotational}} = \frac{1}{2} I (\omega_2^2 - \omega_1^2) \)

Power, \( P = \tau \omega \)

19. \( \vec{L} = I_1 \vec{\omega} \)

20. \( \vec{\tau}_{\text{external}} = \frac{d\vec{L}}{dt} \)

\( \vec{L} \) is conserved if \( \tau_{\text{external}} = 0 : I_1 \omega_1 = I_2 \omega_2 \)
1. courses.lumenlearning.com/suny-osuniversityphysics/chapter/10-1-rotational-variables/
   (Also follow the NEXT button for subsequent 7 webpages)

2. Video lectures:
   On rotating objects, moments of inertia, rotational KE:
   youtube.com/watch?v=fDJeVR0o__w
   ocw.mit.edu/courses/physics/8-01sc-classical-mechanics-fall-2016/week-10-rotational-motion/

   On angular momentum and torque:
   youtube.com/watch?v=sNaaL19opxw
   ocw.mit.edu/courses/physics/8-01sc-classical-mechanics-fall-2016/week-11-angular-momentum/

   On rolling motion:
   youtube.com/watch?v=XPuF__dECVI
   ocw.mit.edu/courses/physics/8-01sc-classical-mechanics-fall-2016/week-12-rotations-and-translation-rolling/
## CHAPTER OUTLINE

<table>
<thead>
<tr>
<th>Exercises</th>
<th>Page No.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1 Introduction</td>
<td>93</td>
</tr>
<tr>
<td>2.2 Fluid, Fluids at rest</td>
<td>93</td>
</tr>
<tr>
<td>2.3 Pressure</td>
<td>93</td>
</tr>
<tr>
<td>2.3.1 Pressure due to a liquid column</td>
<td>93</td>
</tr>
<tr>
<td>2.3.2 Atmospheric pressure</td>
<td>93</td>
</tr>
<tr>
<td>2.3.3 Absolute pressure and Gauge pressure</td>
<td>93</td>
</tr>
<tr>
<td>2.3.4 Hydrostatic Paradox</td>
<td>93</td>
</tr>
<tr>
<td>2.3.5 Pascal’s law, Applications</td>
<td>93</td>
</tr>
<tr>
<td>2.3.6 Measurement of pressure</td>
<td>93</td>
</tr>
<tr>
<td>2.4 Surface tension</td>
<td>103</td>
</tr>
<tr>
<td>2.4.1 Molecular theory of surface tension</td>
<td>103</td>
</tr>
<tr>
<td>2.4.2 Surface tension and Surface energy</td>
<td>103</td>
</tr>
<tr>
<td>2.4.3 Angle of contact</td>
<td>109</td>
</tr>
<tr>
<td>2.4.4 Effect of impurity and temperature on surface tension</td>
<td>109</td>
</tr>
<tr>
<td>2.4.5 Excess pressure across the free surface of a liquid</td>
<td>109</td>
</tr>
<tr>
<td>2.4.6 Explanation of formation of drops and bubbles</td>
<td>109</td>
</tr>
<tr>
<td>2.4.7 Capillary action</td>
<td>116</td>
</tr>
<tr>
<td>2.5 Fluids in motion</td>
<td>121</td>
</tr>
<tr>
<td>2.6 Critical velocity and Reynolds number</td>
<td>121</td>
</tr>
<tr>
<td>2.6.1 Viscosity</td>
<td>121</td>
</tr>
<tr>
<td>2.6.2 Coefficient of viscosity</td>
<td>121</td>
</tr>
<tr>
<td>2.7 Stokes’ Law</td>
<td>121</td>
</tr>
<tr>
<td>2.7.1 Terminal velocity</td>
<td>121</td>
</tr>
<tr>
<td>2.8 Equation of continuity</td>
<td>129</td>
</tr>
<tr>
<td>2.9 Bernoulli equation, Applications</td>
<td>132</td>
</tr>
<tr>
<td>Multiple Choice Questions</td>
<td>139</td>
</tr>
<tr>
<td>Formulae at a Glance</td>
<td>142</td>
</tr>
<tr>
<td>Problems for Practice</td>
<td>143</td>
</tr>
<tr>
<td>Memory map</td>
<td>147</td>
</tr>
<tr>
<td>Internet my friend</td>
<td>148</td>
</tr>
</tbody>
</table>
Q. 1. What is a fluid? Give two examples. (2 marks)
Ans. A fluid is a substance that can flow. A fluid has shear modulus $\approx 0$ and yield to shear. Under a shear stress and a pressure gradient, a fluid begins to flow. Liquids, gases and plasmas are collectively called fluids.

Examples: All gases, all liquids, molten glass and lava, honey, etc.

Do you know? (Textbook page 26)
Plasma is a phase of matter which exists at very high temperatures, at which molecules may dissociate into atoms and ions, and further into electrons and protons. Plasma, however, has very different properties from the three other common phases of matter (viz., solid, liquid and gas) due to the strong electrical forces between the charges.

Q. 2. What is an ideal fluid? OR State the characteristics of an ideal fluid. (2 marks)
Ans. An ideal fluid is one that has the following properties:

1. It is incompressible, i.e., its density has a constant value throughout the fluid.
2. Its flow is irrotational, i.e., the flow is steady or laminar. In an irrotational flow, the fluid doesn’t rotate like in a whirlpool and the velocity of the moving fluid at a specific point doesn’t change over time. (Many fluids change from laminar to turbulent flow as the speed of the fluid increases above some specific value. This can dramatically change the properties of the fluid.)

Q. 3. What is an incompressible fluid? (1 mark)
Ans. An incompressible fluid is one which does not undergo change in volume for a large range of pressures. Thus, its density has a constant value throughout the fluid. In most cases, all liquids are incompressible.

Q. 4. How does a fluid differ from a solid? (2 marks)
Ans. In response to a shear as well as normal force, a solid deforms and develops a restoring force. Within the elastic limit, both types of deformation is reversible. A solid changes its shape under a shear. A normal force causes a change in its length or volume. If the elastic limit is exceeded, the solid gets an irreversible deformation called a permanent set.

A fluid, on the other hand, can only be subjected to normal compressive stress, called pressure. A fluid does not have a definite shape, so that under a shear it begins to flow. Real fluids, with non-zero viscosity, display a weak resistance to shear.

Q. 5. State the properties of a fluid. (1/2 mark each)
Ans. Properties of a fluid:

1. They do not resist deformation and get permanently deformed.
2. They are capable of flowing.
3. They take the shape of the container.

Remember this (Textbook page 27)
The term fluid includes both the liquid and gas phases. It is commonly used, as a synonym for liquid only, without any reference to gas. For example, ‘brake fluid’ is hydraulic oil and will not perform its required function if there is gas in it! This colloquial use of the term is also common in the fields of medicine and nutrition, e.g., “take plenty of fluids.”

Q. 6. Define pressure. State its SI and CGS units and dimensions. (3 marks)
Ans. **Definition:** The pressure at a point in a fluid in hydrostatic equilibrium is defined as the normal force per unit area exerted by the fluid on a surface of infinitesimal area containing the point.
Thus, the pressure, \( p = \lim_{\Delta A \to 0} \frac{F}{\Delta A} \)

where \( F \) is the magnitude of the normal force on a surface of area \( \Delta A \). The pressure is defined to be a scalar quantity.

**SI unit**: the pascal (Pa), \( 1 \text{ Pa} = 1 \text{ N} \cdot \text{m}^{-2} \)

**CGS unit**: the dyne per square centimetre (dyn/cm\(^2\))

**Dimensions**: \([p] = [F][A^{-1}] = [MLT^{-2}, L^{-2}] = [ML^{-1}T^{-2}]\)

Q. 7. State two non-SI units of pressure. \((1 \text{ mark})\)

**Ans.** Two non-SI units, which are either of historical interest, or are still used in specific fields are the bar and the torr.

1 bar = 0.1 MPa = 100 kPa = 1000 hPa = 10\(^5\)Pa
1 torr = (101325/760) Pa = 133.32 Pa

**Note**: Their use in modern scientific and technical work is strongly discouraged.

Q. 8. If a force of 200 N is applied perpendicular to a surface of area 10 cm\(^2\), what is the corresponding pressure? \((1 \text{ mark})\)

**Ans.** Pressure, \( p = \frac{F}{A} = \frac{200 \text{ N}}{10 \times 10^{-4} \text{ m}^2} = 2 \times 10^7 \text{ N/m}^2\).

Q. 9. Explain why the forces acting on any surface within a fluid in hydrostatic equilibrium must be normal to the surface. \((2 \text{ marks})\)

**Ans.** In a fluid, the molecules are in a state of random motion and the intermolecular cohesive forces are weak. If a fluid is subjected to a tangential force (shear) anywhere within it, the layers of the fluid slide over one another, i.e., the fluid begins to flow. Thus, a fluid cannot sustain a tangential force. So, in turn, a fluid at rest cannot exert a tangential force on any surface with which it is in contact. It can exert only a force normal to the surface. Hence, if a fluid is in hydrostatic equilibrium (i.e., at rest), the force acting on any surface within the fluid must be normal to the surface.

**Can you tell?**

**(Textbook page 27)**

**Why does a knife have a sharp edge or a needle has a sharp tip?**

For a given force, the pressure over which the force is exerted depends inversely on the area of contact; smaller the area, greater the pressure. For instance, a force applied to an area of 1 mm\(^2\) applies a pressure that is 100 times as great as the same force applied to an area of 1 cm\(^2\). The edge of a knife or the tip of a needle has a small area of contact. That is why a sharp needle is able to puncture the skin when a small force is exerted, but applying the same force with a finger does not.

**Use your brain power**

**(Textbook page 27)**

A student of mass 50 kg is standing on both feet. Estimate the pressure exerted by the student on the Earth. Assume reasonable value to any quantity you need; justify your assumption. You may use \( g = 10 \text{ m/s}^2 \), By what factor will it change if the student lies on back?

Assume area of each foot = area of a 6 cm \( \times \) 25 cm rectangle.

\[ \therefore \text{Area of both feet} = 0.03 \text{ m}^2 \]

\[ \therefore \text{The pressure due to the student’s weight} \]

\[ = \frac{mg}{A} = \frac{50 \times 10}{0.03} = 16.7 \text{ kPa} \]

According to the most widely used Du Bois formula for body surface area (BSA), the student’s BSA = 1.5 m\(^2\), so that the area of his back is less than half his BSA, i.e., < 0.75 m\(^2\). When the student lies on his back, his area of contact is much smaller than this. So, estimating the area of contact to be 0.3 m\(^2\), i.e., 10 times more than the area of his feet, the pressure will be less by a factor of 10 or more, [Du Bois formula : BSA = 0.2025 \( \times \) \( W^{0.425} \) \( \times \) \( H^{0.725} \), where \( W \) is weight in kilogram and \( H \) is height in metre.]
Q. 10. Would you rather have someone wearing studs step on your foot or have someone wearing tennis shoes step on your foot? (1 mark)

Ans. A person would exert the same downward force regardless of whether he or she was wearing studs or tennis shoes. However, if the person were wearing studs, the force would be applied over a much smaller area, so the pressure would be greater (and so would be more painful).

Q. 11. Would you rather have an elephant stand on your foot directly or have an elephant balance on a thumbtack on top of your foot? (1 mark)

Ans. The downward force of the elephant’s weight would be applied over a much smaller area if it were balancing on a thumbtack, so the pressure would be greater.

Remember this
(Textbook page 28)
The concept of pressure is useful in dealing with fluids, i.e., liquids and gases. As fluids do not have definite shape and volume, it is convenient to use the quantities pressure and density rather than force and mass when studying hydrostatics and hydrodynamics.

Q. 12. Derive an expression for pressure exerted by a liquid column. (3 marks)

Ans. At a point at depth \( h \) below the surface of a liquid of uniform density \( \rho \), the pressure due to the liquid is due to the weight per unit area of a liquid column of height \( h \) above that point.

In Fig. 2.2, to find the pressure due to the liquid at point \( P \), consider the cylindrical liquid column, of cross section \( A \) and height \( h \), above that point.

The weight of this liquid column

\[ = \text{volume} \times \text{density} \times \text{acceleration due to gravity} \]

\[ = (Ah)(\rho)(g) \]

\[ \therefore \text{Pressure due to the liquid at depth} \ h \]

\[ = \frac{\text{weight of the liquid column}}{\text{cross sectional area}} \]

\[ = \frac{Ah\rho g}{A} = h\rho g \]

If the free surface of the liquid is open to the atmosphere, the pressure on the surface is the atmosphere pressure \( p_0 \). Then, the absolute pressure within the liquid at a depth \( h \) is

\[ p = p_0 + h\rho g \]

Q. 13. State the characteristics of pressure due to a liquid at rest at a point within it. (1/2 mark each)

Ans. Characteristics of pressure due to a liquid at rest at a point within it:

1. Within a liquid of constant density, the pressure is directly proportional to the depth.
2. At the same depth within liquids of different densities, the pressure is directly proportional to the density of the liquid.
3. Within a liquid of constant density, the pressure at a given depth is directly proportional to the acceleration due to gravity.
4. The pressure at a point within a given liquid is the same in all directions.
5. The pressure at all points at the same horizontal level within a given liquid is the same.

Q. 14. How much force is exerted on one side of an 8.50 cm by 11.0 cm sheet of paper by the atmosphere? How can the paper withstand such a force? (2 marks)

Ans. Pressure \( p = F/A \). Therefore, the force on one side is

\[ F = pA = (1.013 \times 10^5 \text{ Pa}) (8.50 \times 11.0 \times 10^{-4} \text{ m}^2) = 947.2 \text{ N} \]

The pressure at a point within a fluid being the same in all directions, the same force acts on the other side of the paper. Thus, the net force on the paper is zero.
Q. 15. What is the pressure exerted by a water column of height 1 m? \( [\rho = 10^3 \text{ kg/m}^3, g = 9.8 \text{ m/s}^2] \)

\[ 1 \text{ m} (10^3 \text{ kg/m}^3) \times (9.8 \text{ m/s}^2) \]

\[ = 9.8 \times 10^3 \text{ Pa} \]

**Ans.** Pressure exerted by the water column

\( h \),

\( p = h \rho g \)

\( = 1 \text{ m} (10^3 \text{ kg/m}^3) \times (9.8 \text{ m/s}^2) \)

\( = 9.8 \times 10^3 \text{ Pa} \)

**Remember this**

(Textbook page 28)

1. As \( p = h \rho g \), the pressure exerted by a fluid at rest is independent of the shape and size of the container.

2. \( p = h \rho g \) is true for liquids as well as for gases.

**Q. 16.** Would you rather breathe through a 2 m long tube to the surface in 1.5 m of water in the ocean or breathe at the beach near the ocean? (1 mark)

**Ans.** The pressure on one’s lungs would be much greater under water than standing on the beach because the force exerted by the water on the lungs is greater than the force exerted by the air. Because the pressure of the water on the lungs is so much greater than the outward pressure of the air inside, it would be difficult to take a breath under 1.5 m of water than on the beach.

**Q. 17.** What is atmospheric pressure? Define standard atmospheric pressure. (2 marks)

**Ans.** The Earth’s surface is covered with a layer of atmosphere, with more than 99% of the atmosphere lying within 31 km of the surface. The weight of the atmosphere exerts a downward thrust on any surface lying within it. This gives rise to atmospheric pressure. The atmospheric pressure at any height above the Earth’s surface is the weight of a column of air of unit cross section from that altitude to the top of the atmosphere.

**Definition:** Standard atmospheric pressure, or *one atmosphere* of pressure, is defined as the pressure equivalent of a column of mercury that is exactly 0.7600 m in height at 0 °C.

We can calculate this equivalent pressure in SI unit by using the density of mercury

\[ \rho = 13.6 \times 10^3 \text{ kg/m}^3 \text{ and } g = 9.80 \text{ m/s}^2. \]

\[ 1 \text{ atm} = (0.76 \text{ m}) (13.6 \times 10^3 \text{ kg/m}^3) (9.80 \text{ m/s}^2) \]

\[ = 1.013 \times 10^5 \text{ Pa} = 101.3 \text{ kPa} \]

[Note: 1000 mbar = 100 kPa. Therefore, 1 atm = 1013 mbar.]

**Q. 18.** Explain gauge pressure and absolute pressure within a liquid open to the atmosphere. OR Explain the effect of gravity on fluid pressure. (3 marks)

**Ans.** Consider a cylindrical fluid column of uniform density \( \rho \), area of cross section \( A \) and height \( h \), Fig. 2.3

The mass of the fluid within the column is

\[ m = \text{density} \times \text{volume} \]

\[ = \rho Ah \]

If \( p_1 \) and \( p_2 \) are the pressures at the top and bottom faces of the column, the forces on the top and bottom faces are respectively.

\[ F_1 = p_1A + mg \quad \text{(downward)} \]

and \( F_2 = p_2A \quad \text{(upward)} \)

If the column is in equilibrium,

\[ F_2 = F_1 \]

\[ \therefore p_2A = p_1A + mg = p_1A + \rho Ahg \]

\[ \therefore p_2 - p_1 = \rho hg \]

If \( p_1 = p_0 \) = atmospheric pressure, the **gauge pressure**

\[ p_2 - p_0 = \rho hg \]

In the absence of gravity, \( p_2 = p_0 \). But since atmospheric pressure is equal to the weight per unit area of the entire air column above, even \( p_0 \) will be zero in the absence of gravity.

**Q. 19.** Define gauge pressure. (1 mark)

**When is gauge pressure (i) positive (ii) negative?** (1 mark)

**Give two examples where gauge pressure is more relevant.** (1 mark)

**Ans.** **Definition:** Gauge pressure is the pressure exerted by a fluid relative to the local atmospheric pressure.
Gauge pressure, \( p_g = p - p_0 \)
where \( p \) is the absolute pressure and \( p_0 \) is the local atmospheric pressure.

When the pressure inside a closed container or tank is greater than atmospheric pressure, the pressure reading on a pressure gauge is positive. The pressure inside a ‘vacuum chamber’—a rigid chamber from which some of the air is pumped out—is less than the atmospheric pressure, so a pressure gauge on the chamber designed to measure negative pressure reads a negative value.

At a depth within a liquid of density \( \rho \), the gauge pressure is
\[
   p_g = p - p_0 = h \rho g
\]

**Examples** : Tyre pressure gauge, blood pressure gauge, pressure gauge on an oxygen or scuba tank.

Q. 20. Define absolute pressure. (1 mark)

Ans. **Definition** : The absolute pressure, or total pressure, is measured relative to absolute zero on the pressure scale—which is a perfect vacuum—and is the sum of gauge pressure and atmospheric pressure. It is the same as the thermodynamic pressure.

Absolute pressure accounts for the atmospheric pressure, which in effect adds to the pressure in any fluid which is not enclosed in a rigid container i.e., the fluid is open to the atmosphere.

\[
   p = p_0 + p_g
\]
where \( p_0 \) and \( p_g \) are respectively atmospheric pressure and the gauge pressure.

Absolute pressure can be never negative.

Q. 21. If your tyre gauge reads 2.31 atm (234.4 kPa), what is the absolute pressure? (1 mark)

Ans. The absolute pressure, \( p = p_0 + p_g = 1 \text{ atm} + 2.31 \text{ atm} = 3.31 \text{ atm} \approx 335 \text{ kPa} \).

Q. 22. State and explain the hydrostatic paradox. OR Explain hydrostatic paradox. (3 marks)

Ans. **Hydrostatic paradox** : The normal force exerted by a liquid at rest on the bottom of the containing vessel is independent of the amount of liquid or the shape of the container, but depends only on the area of the base and its depth from the liquid surface.

Consider several vessels of the same base area as shown in Fig. 2.5(a). A liquid is poured into them to the same level, so that the pressure is the same at the bottom of each vessel. Then it must follow that the normal force on the base of each vessel is also the same. However, when placed on a scale balance they are found to have different weights. Herein lies the paradox.

**Explanation** : Since a liquid always exerts a normal force on a wall of the container, in turn, the wall exerts an equal and opposite reaction on the liquid. In the case of tube A, this reaction is everywhere horizontal; so that the normal force at the base of A is only due to the weight of the liquid column above.

The reaction of the slanted wall of vessel C has a vertical component, as shown in Fig. 2.5(b), which supports the weight of the liquid above the slanted side. Hence, the normal force at the base of C is only due to the weight of the vertical liquid column above the base, shown by dashed lines. Since the vessels A and C are filled to the same height and have the same base area, the pressures at the bases of the two vessels are also same. However, the volume of the liquid being clearly different, they have different weights.
In the case of vessel B, the downward vertical component of the reaction of the wall provides an extra normal force at the base, as shown in Fig. 2.5 (c).

Q. 23. Can pressure in a fluid be increased by pushing directly on the fluid? Give an example. (1 mark)

Ans. Yes, but it is much easier if the fluid is enclosed.

Examples: (1) The heart increases the blood pressure by pushing on the blood in an enclosed ventricle. (2) Hydraulic brakes, lifts and cranes operate by pushing on oil in an enclosed system.

Q. 24. State Pascal’s law. (1 mark)

Ans. Pascal’s law: A change in the pressure applied to an enclosed fluid at rest is transmitted undiminished to every point of the fluid and to the walls of the container, provided the effect of gravity can be ignored.

[Note: The law does not say that ‘the pressure is the same at all points of a fluid’ – rightly so, since the pressure in a fluid near Earth varies with height. Rather, the law applies to the change in pressure. According to Pascal’s law, if the pressure on an enclosed static fluid is changed by a certain amount, the pressure at all points within the fluid changes by the same amount.]

The above law is due to Blaise Pascal (1623–62), French mathematician and physicist.

Q. 25. Describe an experimental proof of Pascal’s law. (1 mark)

Ans. Consider a spherical vessel having four cylindrical tubes A, B, C and D each fitted with air-tight frictionless pistons of areas of cross section $A$, $A/2$, $2A$ and $3A$, respectively, as shown in Fig. 2.6. The vessel is filled with an incompressible liquid such that there is no air between the liquid and the pistons.

If the piston A is pushed with a force $F$, the pressure on the piston and the liquid in the vessel is $p_A = F/A$. It is seen that the other three pistons are pushed outwards. To keep these pistons at their respective original positions, forces of $F/2$, $2F$ and $3F$, respectively are required to be applied on pistons B, C and D respectively to hold them. Then, the pressures on the respective pistons are

$$p_B = \frac{F/2}{A/2} = F/A, \quad p_C = \frac{2F}{2A} = F/A, \quad \text{and} \quad p_D = \frac{3F}{3A} = F/A$$

This indicates that the pressure applied is transmitted equally to all parts of liquid. This proves Pascal law.

Q. 26. Explain the principle of multiplication of thrust. (3 marks)

Ans. Principle of multiplication of thrust by transmission of fluid pressure: The normal force exerted by a fluid on any surface in contact with it is called the thrust. Consider two hydraulically connected cylinders, one of cross section $a$ and the other $A$, as in Fig. 2.6. If a force $F_a$ is exerted on the smaller piston, pressure $p = \frac{F_a}{a}$ is produced and transmitted undiminished throughout the liquid. Then, the thrust $F_A$ on the larger piston is

$$F_A = pA = \frac{A}{a} F_a$$

If $A = na$, $F_A = nF_a$, i.e., the thrust on the larger piston is multiplied $n$ times. This is known as the principle of multiplication of thrust by transmission of fluid pressure.

Q. 27. State any two applications of Pascal’s law. (1 mark)

Briefly explain their working. (3 marks each)

Ans. Applications of Pascal’s law:

(1) Hydraulic car lift and hydraulic press

(2) Hydraulic brakes.
All the above applications work on the principle of multiplication of thrust by transmission of fluid pressure.

(1) **Working of a hydraulic lift** (Fig. 2.7) : Two hydraulically connected cylinders, one of cross section \( a \) and the other \( A \), are such that \( A \) is many times larger than \( a : A = na \). If a force \( F_a \) is exerted on the smaller piston, a pressure \( p = \frac{F_a}{a} \) is produced and transmitted undiminished throughout the liquid. Then, the thrust \( F_A \) on the larger piston

\[
F_A = pA = \frac{A}{a} F_a = nF_a
\]

is \( n \) times greater than that on the smaller piston. A platform attached to the larger piston can lift a car (as in a hydraulic car lift), or press bales of cotton or paper against a fixed rigid frame (as in Brahma’s.)

(2) **Working of hydraulic brakes in a car** (Fig 2.8) : Brakes which are operated by means of hydraulic pressure are called hydraulic brakes. An automobile hydraulic brake system, shown schematically in Fig. 2.8, has fluid-filled master and slave cylinders connected by pipes. When the brake pedal is pushed, it depresses the piston of the pedal or master cylinder through a lever. The change in pressure in the master cylinder is transmitted to the four wheel or slave cylinders. Since the brake fluid is incompressible, the pistons of the slave cylinders are pushed out, pressing braking pads onto the braking discs on the wheels. Note that we can add as many wheel cylinders as we wish.

The master cylinder has a much smaller area of cross section \( A_m \) compared to the combined area \( A_s \) of the slave cylinders. Hence, with a small force \( F_m \) on the master cylinder, a force \( F_s = \frac{A_s}{A_m} F_m \) which is greater than \( F_m \) is applied on each slave cylinder. Consequently, the master piston has to travel several inches to move the slave pistons the fraction of an inch it takes to apply the brakes. But the arrangement allows great force to be exerted at the brake pads.

**Notes** : (1) Pascal’s law laid the foundation for hydraulics, the use of a liquid under pressure to transfer force or motion, or to increase an applied force. It is one of the most important branches in modern engineering.

(2) A hydraulic system, as an example of a simple machine, can increase force but cannot do more work than is done on it. Work being force times the distance moved, the piston in a wheel cylinder moves through a smaller distance than that in the pedal cylinder. Power brakes in modern automobiles have a motorized pump that does most of the work in the system.

Q. 28. Why are liquids used in hydraulic systems but not gases ?

(2 marks)

**Ans.** Liquids are used in a hydraulic system because liquids are incompressible and transmit a change in pressure undiminished to all parts of the system. On the other hand, on increasing the pressure, a gas will be compressed into a smaller volume due to which there will be no transmission of force or motion.

Q. 29. State one advantage of hydraulic brakes in an automobile.

(2 marks)

**Ans.** Advantages of hydraulic brakes:

(1) By Pascal’s law, equal braking effort is applied to all the wheels.
(2) It is easily possible to increase or decrease the applied force—during the design stage—by changing the size of piston and cylinder relative to other.

Q. 30. What is a barometer? Explain the use of a simple mercury barometer to measure atmospheric pressure. (3 marks)

Ans. A barometer is an instrument to measure atmospheric pressure. The mercury barometer was invented by Evangelista Torricelli (1609–47). Italian physicist and mathematician.

![Fig. 2.9: Torricelli’s mercury barometer](image)

A strong glass tube, about one metre long and closed at one end, is filled with mercury. With a finger over the open end, the tube is inverted and the open end is immersed into a bowl of mercury. When the finger is removed, the mercury level in the tube drops. The mercury column in the tube stands at a height \( h \) for which the pressure at point \( A \) inside the tube due to the weight of the mercury column is equal to the atmospheric pressure \( p_0 \) outside (at point \( B \)).

The space at the closed end of the tube, after the mercury level drops, is nearly a vacuum, known as the Torricellian vacuum, so the pressure there can be taken as zero. It, therefore, follows that \( p_0 = pgh \)

Where \( p \) is the density of mercury and \( h \) is the height of the mercury column.

Q. 31. What is an open tube manometer? Briefly describe its function with a neat diagram. (3 marks)

Ans. An open tube manometer is a device to measure the pressure of a gas in a vessel. It consists of a U-shaped tube containing a liquid (say, mercury) of density \( p \), as shown in Fig. 2.10.

One end of the tube is connected to the vessel while the other end is open to the atmosphere. The pressure \( p \) at point \( A \) is the (unknown) pressure of the gas in the vessel. The pressure on the mercury column in the open tube is the atmospheric pressure \( p_0 \).

![Fig. 2.10: An open tube manometer](image)

A point \( B \), at the same horizontal level as \( A \), is at a depth \( h \) from the surface of mercury in the open tube. Therefore, the pressure at \( B \) is \( p_0 + pgh \).

The pressures at points \( A \) and \( B \) at the same liquid level being the same, equating the unknown pressure \( p \) (at \( A \)) to the pressure at \( B \).

\[ p = p_0 + pgh \]

The pressure \( p \) is called the absolute pressure, and the difference in pressure \( p - p_0 \) is called the gauge pressure.

★ Q. 32. Why is a low density liquid used as a manometric liquid in a physics laboratory? (2 marks)

Ans. An open tube manometer measures the gauge pressure, \( p - p_0 = hpg \), where \( p \) is the pressure being measured, \( p_0 \) is the atmospheric pressure, \( h \) is the difference in height between the manometric liquid of density \( p \) in the two arms. For a given pressure \( p \), the product \( hp \) is constant. That is, \( p \) should be small for \( h \) to be large. Therefore, for noticeably large \( h \), laboratory manometer uses a low density liquid.
Q. 33. An open tube manometer is connected to (i) a vacuum-packed candy jar, with the atmospheric pressure in the open tube supporting a column of fluid of height $h$ (ii) a gas tank, with the absolute pressure in the tank supporting a column of fluid of height $h$. Is the absolute pressure in the jar and the gas tank greater than or less than the atmospheric pressure? By how much? (2 marks)

Ans. In the first case, $p_{abs}$ is less than the atmospheric pressure, whereas in the second case, $p_{abs}$ is greater than the atmospheric pressure. In both cases, $p_{abs}$ differs from the atmospheric pressure by the gauge pressure $\frac{h}{g}$, where $\rho$ is the density of the fluid in the manometer.

Fig. 2.11: Mercury manometer connected to (a) a vacuum packed jar (b) a gas tank (For reference only)

Solved Problems 2.1–2.3.6

Q. 34. Solve the following:
★ (1) Find the pressure 200 m below the surface of the ocean if the pressure on the free surface of liquid is one atmosphere. [Density of sea water = 1060 kg/m$^3$] (2 marks)

Solution:
Data: $h = 200$ m, $p = 1060$ kg/m$^3$, $g = 9.8$ m/s$^2$

Absolute pressure,

$p = p_0 + \rho gh$

$= (1.013 \times 10^5) + (20)(1060)(9.8)$

$= 21.789 \times 10^4 = 2.1789$ MPa

(2) For diver’s safety, a 10 m platform diving pool should be 5 m deep. However, with an excellent dive, a diver usually reaches a maximum depth of 2.5 m. (i) Calculate the pressure due to the weight of the water at the depth of 2.5 m. (ii) Calculate the depth below the surface of water at which the pressure due to the weight of the water equals 1.0 atm. [Density of water = $10^3$ kg/m$^3$, 1 atm = 101.3 kPa] (3 marks)

Solution:
Data: $h = 250$ m, $\rho = 1000$ kg/m$^3$, $g = 9.8$ m/s$^2$,
1 atm = 101.3 kPa

(i) $p = \rho gh = (250)(1000)(9.8) = 24.18$ atm

This gives the pressure at a depth of 250 m.

(ii) $h = \frac{p}{\rho g} = \frac{1.013 \times 10^5}{10^3 \times 9.8} = 10.34$ m

This gives the required depth.

(3) Suppose a dam is 250 m wide and the water is 40 m deep at the dam. What is (i) the average pressure on the dam (ii) the force exerted against the dam due to the water? (3 marks)

Solution:
Data: Width, $L = 250$ m, depth $H = 40$ m, $\rho = 1000$ kg/m$^3$, $g = 9.8$ m/s$^2$

Since pressure increases linearly with depth, the average pressure $p_{av}$ due to the weight of the water is the pressure at the average depth $h$ of 20 m. The force exerted on the dam by the water is the average pressure times the area of contact, $F = p_{av}A = p_{av}LH$.

(i) $p_{av} = \rho gh = (20)(1000)(9.8) = 1.96 \times 10^5$ Pa

(ii) $F = p_{av}A = p_{av}LH = (1.96 \times 10^5)(250)(40) = 1.96 \times 10^9$ N

(4) A car lift at a service station has a piston of diameter 30 cm. The lift and piston weigh 800 kg wt. What pressure (in excess of the atmospheric pressure) must be exerted on the piston to raise a car weighing 1700 kg wt at a constant speed? [$g = 9.8$ m/s$^2$] (3 marks)

2. MECHANICAL PROPERTIES OF FLUIDS
Solution:

Data: Piston diameter, \( D = 30 \text{ cm} = 0.3 \text{ m} \), mass of lift and piston, \( m = 800 \text{ kg} \), mass of car, \( M = 1700 \text{ kg} \)

Cross-sectional area of the piston.
\[
A = \frac{\pi D^2}{4} = \frac{3.142 \times (0.3 \text{ m})^2}{4} = 7.07 \times 10^{-2} \text{ m}^2
\]

Total weight of the car and lift,
\[
W = (m + M)g = (800 \text{ kg} + 1700 \text{ kg}) \times (9.8 \text{ m/s}^2) = 2.45 \times 10^4 \text{ N}
\]

Therefore, the pressure on the piston
\[
p = \frac{F}{A} = \frac{W}{A} = \frac{2.45 \times 10^4 \text{ N}}{7.07 \times 10^{-2} \text{ m}^2} = 3.465 \times 10^5 \text{ Pa}
\]

A pressure of \( 3.465 \times 10^5 \text{ Pa} \) must be exerted on the piston.

(5) The diameters of two pistons in a hydraulic press are 5 cm and 25 cm respectively. A force of 20 N is applied to the smaller piston. Find the force exerted on the larger piston. (2 marks)

Solution:

Data: \( D_1 = 5 \text{ cm}, D_2 = 25 \text{ cm}, F_1 = 20 \text{ N} \)

By Pascal’s law,
\[
\frac{F_1}{A_1} = \frac{F_2}{A_2}
\]

\( \therefore \) the force on the larger piston is
\[
F_2 = F_1 \frac{A_2}{A_1} = F_1 \frac{\pi D_2^2/4}{\pi D_1^2/4} = F_1 \left( \frac{D_2}{D_1} \right)^2 = 20 \text{ N} \times 5^2 \left( \frac{25}{5} \text{ cm} \right)^2 = 20 \times 25 = 500 \text{ N}
\]

(6) In a hydraulic lift, the input piston has surface area of 30 cm\(^2\) and the output piston has a surface area of 1500 cm\(^2\). If a force of 25 N is applied to the input piston, calculate the force on the output piston. (2 marks)

Solution:

Data: \( A_1 = 30 \text{ cm}^2 = 3 \times 10^{-3} \text{ m}^2 \), \( A_2 = 1500 \text{ cm}^2 = 0.15 \text{ m}^2 \), \( F_1 = 25 \text{ N} \)

By Pascal’s law,
\[
\frac{F_1}{A_1} = \frac{F_2}{A_2}
\]

\( \therefore \) The force on the output piston,
\[
F_2 = F_1 \frac{A_2}{A_1} = (25) \frac{0.15}{3 \times 10^{-3}} = 25 \times 50 = 1250 \text{ N}
\]

(7) In a hydraulic lift, the input piston has surface area 20 cm\(^2\). The output piston has surface area 1000 cm\(^2\). If a force of 50 N is applied to the input piston, it raises the output piston by 2 m. Calculate the weight of the support on the output piston and the work done by it. (3 marks)

Solution:

Data: \( A_1 = 20 \text{ cm}^2 = 2 \times 10^{-3} \text{ m}^2 \), \( A_2 = 1000 \text{ cm}^2 = 10^{-1} \text{ m}^2 \), \( F_1 = 50 \text{ N}, s_2 = 2 \text{ m} \)

(i) By Pascal’s law,
\[
\frac{F_1}{A_1} = \frac{F_2}{A_2}
\]

\( \therefore \) \( F_2 = F_1 \frac{A_2}{A_1} = (50 \text{ N}) \times \frac{10^{-1} \text{ m}^2}{2 \times 10^{-3} \text{ m}^2} = 50 \times 50 = 2500 \text{ N} \)

This gives the weight of the support on the output piston.

(ii) The work done by the force transmitted to the output piston is
\[
F_2 s_2 = (2500 \text{ N})(2 \text{ m}) = 5000 \text{ J}
\]

(8) A driver pushes the brake pedal of a car exerting a force of 100 N that is increased by the simple lever to a force of 500 N on the pedal (master) cylinder. The hydraulic system transmits this force to the four wheel (slave) cylinders. If the pedal cylinder has a diameter of 0.5 cm and each wheel cylinder has a diameter of 2.5 cm, calculate the magnitude of the force \( F_s \) on each of the wheel cylinder. (2 marks)

Solution:

Data: \( F_m = 500 \text{ N}, D_m = 1 \text{ cm}, D_s = 2.5 \text{ cm} \)

\[
\frac{F_s}{A_s} = \frac{F_m}{A_m}
\]

\( \therefore \) The magnitude of the force on each of the wheel cylinders,
\[
F_s = A_s F_m = \left( \frac{D_s}{D_m} \right)^2 F_m = \left( \frac{2.5}{0.5} \right)^2 (500) = 25 \times 500 = 12.5 \text{ kN}
\]
Mercury manometers are often used to measure arterial blood pressure. The typical blood pressure of a young adult raises the mercury to a height of 120 mm at systolic and 80 mm at diastolic. Express these values in Pascal and bar. [Density of mercury = 13600 kg/m³, 1 mbar = 100 Pa]

Solution:

\[ p_{\text{max}} = p_{\text{syst}} = 120 \text{ mm of Hg}, \quad p_{\text{min}} = p_{\text{dias}} = 80 \text{ mm of Hg}, \quad \rho = 13600 \text{ kg/m}^3, \quad g = 9.8 \text{ m/s}^2, \quad 1 \text{ mbar} = 100 \text{ Pa} \]

\[ p = \rho g h \]

\[ p_{\text{syst}} = (0.120)(1.36 \times 10^4)(9.8) \]

\[ = 1.6 \times 10^4 \text{ Pa} = 16 \text{ kPa} \]

\[ = 1600 \text{ mbar} = 1.5 \text{ bar} \]

And \[ p_{\text{dias}} = (0.08)(1.36 \times 10^4)(9.8) \]

\[ = 1.066 \times 10^4 \text{ Pa} = 10.66 \text{ kPa} \]

\[ = 1066 \text{ mbar} = 1.066 \text{ bar} \]

Q. 35. Describe the phenomenon of surface tension, giving four examples. (3 marks)

Ans. Surface tension is a unique property of liquids that arises because the net intermolecular force of attraction on the liquid molecules at or near a liquid surface differs from that on molecules deep in the interior of the liquid. This results in the tendency of the free surface of a liquid to minimize its surface area and behave somewhat like a stressed elastic membrane.

Surface tension is important in understanding the peculiar behaviour of the free surface of a liquid in many cases as illustrated below:

1. Small quantities of liquids assume the form of spherical droplets, as in mist, or a mercury droplet on a flat surface. This is because the stressed surface ‘skin’ tends to contract and mould the liquid into a shape that has minimum surface area for its volume, i.e., into a sphere.

2. Surface tension is responsible for the spherical shape of freely-falling raindrops and the behaviour of bubbles and soap films.

3. The bristles of a paint brush cling together when it is drawn out of water or paint.

4. A steel needle or a razor blade can, with care, be supported on a still surface of water which is much less dense than the metal from which these objects are made of.

5. Many insects like ants, mosquitoes, water striders, etc., can walk on the surface of water.

Q. 36. Define (1) cohesive force (2) adhesive force. (2 marks)

Give one example in each case. (1 mark)

Ans.

1. **Cohesive force**: The intermolecular force of attraction between two molecules of the same material is called the **cohesive force**.

   **Example**: The force of attraction between two water molecules.

2. **Adhesive force**: The intermolecular force of attraction between two molecules of different materials is called the **adhesive force**.

   **Example**: The force of attraction between a water molecule and a molecule of the solid surface which is in contact with water.

Q. 37. Define (1) range of molecular attraction or molecular range (2) sphere of influence. (2 marks)

Ans.

1. **Range of molecular attraction or molecular range**: Range of molecular attraction or molecular range is defined as the maximum distance between two molecules up to which the intermolecular force of attraction is appreciable.

   [Note: The intermolecular force is a short range force, i.e., it is effective over a very short range – about \(10^{-9}\) m. Beyond this distance, the force is negligible. The intermolecular force does not obey inverse square law.]

2. **Sphere of influence**: The sphere of influence of a molecule is defined as an imaginary sphere with the
molecule as the centre and radius equal to the range of molecular attraction.

**Note:** All molecules lying within the sphere of influence of a molecule are attracted by (as well as attract) the molecule at the centre. For molecules which lie outside this sphere, the intermolecular force due to the molecule at the centre is negligible.

**Q. 38. What is meant by a surface film?** *(1 mark)*

**Ans.** The layer of the liquid surface of thickness equal to the range of molecular attraction is called a **surface film**.

**Q. 39. What is meant by free surface of a liquid?** *(2 marks)*

**Ans.** The surface of a liquid open to the atmosphere is called the **free surface of the liquid**.

**Remember this** *(Textbook page 34)*

While studying pressure, we consider both liquids and gases. But as gases do not have a free surface, they do not exhibit surface tension.

**Q. 40. Explain the phenomenon of surface tension on the basis of molecular theory.** *(3 marks)*

**Ans.** The phenomenon of surface tension arises due to the cohesive forces between the molecules of a liquid. The net cohesive force on the liquid molecules within the surface film differs from that on molecules deep in the interior of the liquid.

Consider three molecules of a liquid: A molecule A well inside the liquid, and molecules B and C lying within the surface film, Fig. 2.12. The figure also shows their spheres of influence of radius R.

(1) The sphere of influence of molecule A is entirely inside the liquid and the molecule is surrounded by its nearest neighbours on all sides. Hence, molecule A is equally attracted from all sides, so that the resultant cohesive force acting on it is zero. Hence, it is free to move anywhere within the liquid.

(2) For molecule B, a part of its sphere of influence is outside the liquid surface. This part contains air molecules whose number is negligible compared to the number of molecules in an equal volume of the liquid. Therefore, molecule B experiences a net cohesive force **downward**.

(3) For molecule C, the upper half of its sphere of influence is outside the liquid surface. Therefore, the resultant cohesive force on molecule C in the downward direction is maximum.

(4) Thus, all molecules lying within a surface film of thickness equal to R experience a net cohesive force directed **into** the liquid.

(5) The surface area is proportional to the number of molecules on the surface. To increase the surface area, molecules must be brought to the surface from within the liquid. For this, work must be done against the cohesive forces. This work is stored in the liquid surface in the form of potential energy. With a tendency to have minimum potential energy, the liquid tries to reduce the number of molecules on the surface so as to have minimum surface area. This is why the surface of a liquid behaves like a stressed elastic membrane.

**Q. 41. Define surface tension.** *(1 mark)*

State its formula and CGS and SI units. *(1 mark)*

**Ans.** The **surface tension** of a liquid is defined as the tangential force per unit length, acting at right angles on either side of an imaginary line on the free surface of the liquid.

If F is the force on one side of a line of length l, drawn on the free surface of a liquid, the surface tension (T) of the liquid is

\[
T = \frac{F}{l}
\]

**The CGS unit of surface tension:** The dyne per centimetre (dyn/cm) or, equivalently, the erg per square centimetre (erg/cm²).
The SI unit of surface tension: The newton per metre (N/m) or, equivalently, the joule per square metre (J/m²).

Q. 42. Obtain the dimensions of surface tension.

Ans. Surface tension is a force per unit length.  
\[ \text{[Surface tension]} = \frac{\text{[force]}}{\text{[length]}} = \frac{[ML^1T^{-2}]}{[M^0L^1T^0]} = [ML^0T^{-2}] \]

OR

Surface tension is also equal to the surface energy per unit surface area of a liquid.

\[ \text{[Surface tension]} = \frac{\text{[energy]}}{\text{[area]}} = \frac{\text{[work]}}{\text{[area]}} = \frac{\text{[force] \times \text{[displacement]}}}{\text{[area]}} = \frac{[ML^1T^{-2}][M^0L^1T^0]}{[M^0L^2T^0]} = [ML^0T^{-2}] \]

Q. 43. Define and explain surface energy of a liquid.

Ans. Surface energy: The surface energy is defined as the extra (or increased) potential energy possessed by the molecules in a liquid surface with an isothermal increase in the surface area of the liquid.

A liquid exerts a resultant cohesive force on every molecule of its surface, trying to pull it into the liquid. To increase the surface area, it is necessary to bring more molecules from inside the liquid to the liquid surface. For this, external work must be done against the net cohesive forces on the molecules. This work is stored in the liquid surface in the form of potential energy.

This extra potential energy that the molecules in the liquid surface have is called the surface energy. Thus, the molecules of a liquid in the surface film possess extra energy.

Dimensions: [surface energy] = [ML²T⁻²]

SI unit: the joule (J).

Q. 44. Why is the surface tension of paints and lubricating oils kept low?

Ans. For better wettability (surface coverage), the surface tension and angle of contact of paints and lubricating oils must be low.

Q. 45. Derive the relation between the surface tension and surface energy of a liquid.

Ans. Suppose a soap film is isothermally stretched over the area enclosed by a U-shaped frame ABCD and a cross-piece PQ that can slide smoothly along the frame, as shown in the figure. Let \( T \) be the surface tension of the soap solution and \( l \), the length of wire PQ in contact with the soap film.

Fig. 2.14: A rectangular soap film

The film has two surfaces, both of which are in contact with the wire. The film tends to contract by exerting a force on wire PQ. As each surface exerts a force \( Tl \), the net force on the wire is \( 2Tl \).

Suppose that wire PQ is pulled outward very slowly through a distance \( dx \) to the position P'Q' by an external force of magnitude \( 2Tl \). The work done by the external force against the force due to the film is

\[ W = \text{applied force} \times \text{displacement} \]
This work is stored in the unit surface area in the form of potential energy. This potential energy is called the surface energy.

Due to the displacement $dx$, the surface area of the film increases. As the film has two surfaces, the increase in its surface area is

$$A = 2ldx$$

Thus, the work done per unit surface area is

$$W = \frac{2Tldx}{2ldx} = T$$

Thus, the surface energy per unit area of a liquid is equal to its surface tension.

**Q. 46. Two soap bubbles of the same soap solution have diameters in the ratio 1 : 2. What is the ratio of work done to blow these bubbles?** (1 mark)

**Ans.** Work done $\propto$ surface area.

$\therefore \frac{W_1}{W_2} = \left(\frac{r_1}{r_2}\right)^2 = (1/2)^2 = 1/4$

$\therefore W_1 : W_2 = 1 : 4.$

**Q. 47. If the surface tension of a liquid is 70 dyn/cm, what is the total energy of the free surface of the liquid drop of radius 0.1 cm?** (1 mark)

**Ans.**

$$E = 4\pi r^2 T = 4 \times \frac{22}{7} \times (0.1)^2 \times 70 = 88 \times 10^{-2} \times 10 = 8.8 \text{ ergs}$$

**Q. 48. The total energy of the free surface of a liquid drop of radius 1 mm is 10 ergs. What is the total energy of a liquid drop (of the same liquid) of radius 2 mm?** (1 mark)

**Ans.**

$$E = 4\pi r^2 T \therefore \frac{E_2}{E_1} = \left(\frac{r_2}{r_1}\right)^2 = \left(\frac{2}{1}\right)^2 = 4$$

$\therefore E_2 = 4E_1 = 4 \times 10 = 40 \text{ ergs}$ is the required energy.

**Q. 49. What is the work done in blowing a soap bubble of radius $r$?** (2 marks)

**Ans.** Let $T$ be the surface tension of a soap solution.

The initial surface area of soap bubble $= 0$

The final surface area of soap bubble $= 2 \times 4\pi r^2$

$\therefore$ The increase in surface area $= 2 \times 4\pi r^2$

The work done in blowing the soap bubble is

$$W = \text{surface tension} \times \text{increase in surface area}$$

$$= T \times 2 \times 4\pi r^2 = 8\pi r^2 T$$

**Q. 50. Why two or more mercury drops form a single drop when brought in contact with each other?** (3 marks)

**Ans.** A spherical shape has the minimum surface area-to-volume ratio of all geometric forms. When two drops of a liquid are brought in contact, the cohesive forces between their molecules coalesces the drops into a single larger drop. This is because, the volume of the liquid remaining the same, the surface area of the resulting single drop is less than the combined surface area of the smaller drops. The resulting decrease in surface energy is released into the environment as heat.

**Proof:** Let $n$ droplets each of radius $r$ coalesce to form a single drop of radius $R$. As the volume of the liquid remains constant,
volume of the drop = volume of $n$ droplets
\[ \frac{4}{3} \pi R^3 = n \times \frac{4}{3} \pi r^3 \]
\[ \therefore R^3 = nr^3 \quad \therefore R = \sqrt[3]{nr} \]
Surface area of $n$ droplets = $n \times 4\pi r^2$
Surface area of the drop = $4\pi R^2 = n^{2/3} \times 4\pi r^2$
\[ \therefore \text{The change in the surface area} \]
\[ = \text{surface area of drop} – \text{surface area of } n \text{ droplets} \]
\[ = 4\pi r^2 (n^{2/3} – n) \]
Since the bracketed term is negative, there is a decrease in surface area and a decrease in surface energy.

**Solved Problems 2.4 – 2.4.2**

Q. 51. Solve the following:

1. Calculate the work done in blowing a soap bubble of radius 4 cm. The surface tension of the soap solution is $25 \times 10^{-3}$ N/m. (2 marks)

**Solution:**

Data:
\[ r = 4 \text{ cm} = 4 \times 10^{-2} \text{ m}, \quad T = 25 \times 10^{-3} \text{ N/m} \]
Initial surface area of soap bubble = 0
Final surface area of soap bubble = $2 \times 4\pi r^2$
\[ \therefore \text{Increase in surface area} = 2 \times 4\pi r^2 \]
The work done
\[ = \text{surface tension} \times \text{increase in surface area} \]
\[ = T \times 2 \times 4\pi r^2 \]
\[ = 25 \times 10^{-3} \times 2 \times 4 \times 3.142 \times (4 \times 10^{-2})^2 \]
\[ = 1.005 \times 10^{-3} \text{ J} \]

2. How much work is required to form a bubble of 2 cm radius from the soap solution having surface tension 0.07 N/m. (2 marks)

**Solution:**

Data: Refer to the solution to Problem (1) above.
The work done = $0.07 \times 8 \times 3.142 \times (2 \times 10^{-2})^2$
\[ = 7.038 \times 10^{-4} \text{ J} \]

3. Two soap bubbles have radii in the ratio 4 : 3. What is the ratio of work done to blow these bubbles? (2 marks)

**Solution:**

Data:
\[ \frac{r_1}{r_2} = \frac{4}{3} \]
Work done, $W = 2TA$
\[ \therefore W_1 = 2T(4\pi r_1^2), \quad W_2 = 2T(4\pi r_2^2) \]

\[ \therefore \frac{W_1}{W_2} = \frac{2T(4\pi r_1^2)}{2T(4\pi r_2^2)} = \left( \frac{r_1}{r_2} \right)^2 \]
\[ = \left( \frac{4}{3} \right)^2 = \frac{16}{9} \]

4. Calculate the work done in increasing the radius of a soap bubble in air from 1 cm to 2 cm. The surface tension of the soap solution is 30 dyn/cm. (2 marks)

**Solution:**

Data:
\[ r_1 = 1 \text{ cm}, \quad r_2 = 2 \text{ cm}, \quad T = 30 \text{ dyn/cm} \]
Initial surface area = $2 \times 4\pi r_1^2$
Final surface area = $2 \times 4\pi r_2^2$
\[ \therefore \text{Increase in surface area} \]
\[ = 2 \times 4\pi r_2^2 – 2 \times 4\pi r_1^2 = 8\pi (r_2^2 – r_1^2) \]
The work done
\[ = \text{surface tension} \times \text{increase in surface area} \]
\[ = T \times 8\pi (r_2^2 – r_1^2) \]
\[ = 30 \times 8 \times 3.142 \times [(2)^2 – (1)^2] \]
\[ = 2262 \text{ ergs} \]

5. A soap film is formed when a rectangular wire frame of area 2 cm x 2 cm is dipped in a soap solution and taken out. If the area of the film is increased to 3 cm x 3 cm, calculate the work done in the process. [Surface tension of the soap film is $3 \times 10^{-2}$ N/m] (2 marks)

**Solution:**

Data:
\[ A_1 = 2 \times 2 \text{ cm}^2 = 4 \times 10^{-4} \text{ m}^2, \quad A_2 = 3 \times 3 \text{ cm}^2 = 9 \times 10^{-4} \text{ m}^2, \quad T = 3 \times 10^{-2} \text{ N/m} \]
As the film has two surfaces, the work done is
\[ W = 2T(A_2 – A_1) \]
\[ = 2[(3 \times 10^{-2})(9 \times 10^{-4} – 4 \times 10^{-4})] \]
\[ = 3.0 \times 10^{-5} \text{ J} = 30 \text{ J} \]

6. A drop of mercury of radius 0.2 cm is broken into 8 droplets of the same size. Find the work done if the surface tension of mercury is 435.5 dyn/cm. (3 marks)

**Solution:**

Let $R$ be the radius of the drop and $r$ be the radius of each droplet.

Data:
\[ R = 0.2 \text{ cm}, \quad n = 8, \quad T = 435.5 \text{ dyn/cm} \]
As the volume of the liquid remains constant, volume of $n$ droplets = volume of the drop
\[ \therefore n \times \frac{4}{3} \pi r^3 = \frac{4}{3} \pi R^3 \]
(7) Calculate the work done when a spherical drop of mercury of radius 2 mm falls from a height and breaks into one million droplets, each of the same size. The surface tension of mercury is 0.5 N/m.

Solution:

Data: \( R = 2 \text{ mm} = 2 \times 10^{-3} \text{ m}, n = 10^{6}, \)
\( T = 0.5 \text{ N/m} \)

Refer to the solution to Problem (6) above.

\[
\therefore \quad r = \frac{R}{\sqrt[n]{n}} = \frac{R}{\sqrt[10^{6}]{10^{6}}} = \frac{R}{10^{2}}
\]

\[
\therefore \quad \text{Increase in surface area} \quad = 4\pi (nr^2 - R^2) \\
= 4\pi \left( 10^{6} \times \frac{R^2}{10^{4}} - R^2 \right) \\
= 4\pi (100 - 1) R^2 \\
= 99 \times 4\pi R^2
\]

\[
\therefore \quad \text{The work done} \\
= 0.5 \times 99 \times 4 \times 3.142 (2 \times 10^{-3})^2 \\
= 2.488 \times 10^{-3} \text{ J}
\]

(8) A mercury drop of radius 0.5 cm falls from a height on a glass plate and breaks into one million droplets, all of the same size. Find the height from which the drop fell. [Density of mercury = 13600 kg/m³, surface tension of mercury = 0.465 N/m] (3 marks)

Solution:

Data: \( R = 0.5 \text{ cm} = 0.5 \times 10^{-2} \text{ m}, n = 10^{6}, \)
\( \rho = 13600 \text{ kg/m}^3, T = 0.465 \text{ N/m}, g = 9.8 \text{ m/s}^2 \)
\[
\frac{4}{3} \pi R^3 = n \times \frac{4}{3} \pi r^3
\]
as the volume of the mercury remains the same.

\[
\therefore \quad r = \frac{R}{\sqrt[n]{n}} = \frac{R}{\sqrt[10^{6}]{10^{6}}} = 0.5 \times 10^{-4} \text{ m}
\]

This gives the radius of a droplet.

By energy conservation, if \( h \) is the height from which the drop of mass \( m \) falls,
\[
mgh = T (4\pi r^3 n - 4\pi R^2)
\]

\[
\therefore \quad \frac{4}{3} \pi R^3 \rho g h = 4\pi T (nr^2 - R^2)
\]

\[
\therefore \quad h = \frac{3T (nr^2 - R^2)}{R^3 \rho g} \\
= \frac{3 \times 0.465 \left[ 10^6 \times (0.5 \times 10^{-4})^2 - (0.5 \times 10^{-2})^2 \right]}{(0.5 \times 10^{-2})^3 \times 13600 \times 9.8} \\
= \frac{3 \times 0.465 \times 0.25 \times (10^{-2} - 10^{-4})}{0.125 \times 0.0136 \times 9.8} \\
= 0.2072 \text{ m}
\]

This gives the required height.

(9) Eight droplets of mercury, each of radius 1 mm, coalesce to form a single drop. Find the change in the surface energy. [Surface tension of mercury = 0.472 J/m²] (3 marks)

Solution:

Data: \( r = 1 \text{ mm} = 1 \times 10^{-3} \text{ m}, T = 0.472 \text{ J/m}^2 \)

Let \( R \) be the radius of the single drop formed due to the coalescence of 8 droplets of mercury.

Volume of 8 droplets = volume of the single drop as the volume of the liquid remains constant.

\[
\therefore \quad 8 \times \frac{4}{3} \pi r^3 = \frac{4}{3} \pi R^3
\]

\[
\therefore \quad 8r^3 = R^3
\]

\[
\therefore \quad 2r = R
\]

Surface area of 8 droplets = \( 8 \times 4\pi r^2 \)
Surface area of single drop = \( 4\pi R^2 \)

\[
\therefore \quad \text{Decrease in surface area} \quad = 8 \times 4\pi r^2 - 4\pi R^2
\]

\[
\quad = 4\pi (8r^2 - R^2) \\
\quad = 4\pi (8r^2 - (2r)^2) \\
\quad = 4\pi \times 4r^2
\]

\[
\therefore \quad \text{The energy released} \\
= \text{surface tension} \times \text{decrease in surface area} \\
= T \times 4\pi \times 4r^2 \\
= 0.472 \times 4 \times 3.142 \times 4 \times (1 \times 10^{-3})^2 \\
= 2.373 \times 10^{-5} \text{ J}
\]

☆ (10) Twenty seven droplets of water, each of radius 0.1 mm, coalesce into a single drop. Find the change in surface energy. [Surface tension of water is 0.072 N/m] (3 marks)
(11) The total energy of the free surface of a liquid drop is \(2 \times 10^{-4} \pi\) times the surface tension of the liquid. What is the diameter of the drop? (Assume all terms in SI unit.) (2 marks)

**Solution:**

**Data :** \(4\pi r^2 T = 2 \times 10^{-4} \pi T\) (numerically)

\[
\therefore 2r^2 = 10^{-4} \\
\therefore r = \frac{10^{-2}}{\sqrt{2}} = 10^{-2} \times 1.414 \\
\therefore d = 2r = 2 \times 0.7072 \times 10^{-2} \\
\therefore d = 1.4144 \times 10^{-2} \text{ m}
\]

This gives the diameter of the liquid drop.

---

**Q. 52.** Define angles of contact. (1 mark)

**Ans.** The angle of contact for a liquid-solid pair (a liquid in contact with a solid) is defined as the angle between the surface of the solid and the tangent drawn to the free surface of the liquid at the extreme edge of the liquid, as measured through the liquid.

**Q. 53.** Draw neat diagrams to show the angle of contact in the case of a liquid which (i) completely wets (ii) partially wets (iii) does not wet the solid. State the characteristics of the angle of contact in each case, giving one example of each. (2 marks each)

**Ans.** Characteristics:

(1) For a liquid, which completely wets the solid, the angle of contact is zero.

For example, pure water completely wets clean glass. Therefore, the angle of contact at the water-glass interface is zero [Fig. 2.16 (a)].

(2) For a liquid which partially wets the solid, the angle of contact is an acute angle.

For example, kerosine partially wets glass, so that the angle of contact is an acute angle at the kerosine-glass interface [Fig. 2.16 (b)].

(3) For a liquid which does not wet the solid, the angle of contact is an obtuse angle.

For example, mercury does not wet glass at all, so that the angle of contact is an obtuse angle at the mercury-glass interface [Fig. 2.16 (c)].

(4) The angle of contact for a given liquid-solid pair is constant at a given temperature, provided the liquid is pure and the surface of the solid is clean.

**Q. 54.** State any two characteristics of angle of contact. (1 mark)

**Ans.** Characteristics of angle of contact:

(1) It depends upon the nature of the liquid and solid in contact, and is constant for a given liquid-solid pair, other factors remaining unchanged.

(2) It depends upon the medium (gas) above the free surface of the liquid.

(3) It is independent of the inclination of the solid to the liquid surface.

(4) It changes with surface tension and, hence, with the temperature and purity of the liquid.

**Q. 55.** Explain why the free surface of some liquids in contact with a solid is not horizontal. OR Explain the formation of concave and convex surface of a liquid on the basis of molecular theory. (3 marks)

**Ans.** For a molecule in the liquid surface which is in contact with a solid, the forces on it are largely the solid-liquid adhesive force \(\vec{F}_A = \vec{PA}\) and the liquid-
liquid cohesive force \( \overrightarrow{F}_C = \overrightarrow{PC} \) \( \overrightarrow{F}_A \) is normal to the solid surface and \( \overrightarrow{F}_C \) is at 45° with the horizontal, Fig. 2.17 (a). The free surface of a liquid at rest is always perpendicular to the resultant \( \overrightarrow{F}_R = \overrightarrow{PR} \) of these forces.

If \( F_C = \sqrt{2F_A} \) \( \overrightarrow{F}_R \) is along the solid surface, the contact angle is 90° and the liquid surface is horizontal at the edge where it meets the solid, as in Fig. 2.17 (a). In general this is not so, and the liquid surface is not horizontal at the edge.

For a liquid which completely wets the solid (e.g., pure water in contact with clean glass), \( F_C \ll F_A \). For a liquid which partially wets the solid (e.g., kerosine or impure water in contact with glass), \( F_C < \sqrt{2F_A} \). If \( F_C \ll F_A \) or if \( F_C < \sqrt{2F_A} \), the contact angle is correspondingly zero or acute and the liquid surface curves up and acquires a concave shape until the tangent PT is tangent to \( \overrightarrow{F}_R \), Fig. 2.17(b).

If \( F_C > \sqrt{2F_A} \) the contact angle is obtuse and the liquid surface curves down, i.e., convex, with the solid.

Q. 56. State the conditions for concavity and convexity of a liquid surface where it is in contact with a solid. (1 mark)

Ans. For a molecule in the liquid surface which is in contact with a solid, the forces on it are largely (i) the solid-liquid adhesive force \( \overrightarrow{F}_A \), normal and into the solid surface and (ii) the liquid-liquid cohesive force \( \overrightarrow{F}_C \) at nearly 45° with the horizontal.

If \( F_C \ll F_A \) or if \( F_C < \sqrt{2F_A} \), the contact angle is correspondingly zero or acute and the liquid surface is concave with the solid.

If \( F_C > \sqrt{2F_A} \), the contact angle is obtuse and the liquid surface curves down, i.e., convex, with the solid.

Q. 57. “The shape of impure water meniscus is concave whereas that of mercury meniscus is convex.” Explain why. OR Explain why the angle of contact of mercury with glass is obtuse while that of water with glass is acute. (3 marks)

Ans. Refer to the answer to Q. 55. (Note: Impure water partially wets glass so that the angle of contact is acute.)

Q. 58. Draw neat labelled diagrams to show angle of contact between (a) pure water and clean glass (b) mercury and clean glass. (2 marks)

Ans.

Can you tell? (Textbook page 38)

How does a waterproofing agent work?

Wettability of a surface, and thus its propensity for penetration of water, depends upon the affinity between the water and the surface. A liquid wets a surface when its contact angle with the surface is acute. A waterproofing coating has angle of contact obtuse and thus makes the surface hydrophobic.
Q. 59. Explain the shape of a liquid drop on a solid surface in terms of interfacial tensions.

OR

Account for the angle of contact in terms of interfacial tensions.

Draw diagram showing force due to surface tension at the liquid-solid, air-solid, air-liquid interface, in case of (i) a drop of mercury on a plane solid surface and (ii) a drop of water on a plane solid surface. Discuss the variation of angle of contact. (3 marks)

Ans. A liquid surface, in general, is curved where it meets a solid. The angle between the solid surface and the tangent to the liquid surface at the extreme edge of the liquid, as measured through the liquid, is called the angle of contact.

![Diagram of liquid drop on solid surface]

Figure 2.19 shows the interfacial tensions that act in equilibrium at the common point of the liquid, solid and gas (air + vapour).

\( T_1 \) = the liquid-solid interfacial tension
\( T_2 \) = the solid-gas interfacial tension
\( T_3 \) = the liquid-gas interfacial tension
\( \theta \) = the angle of contact for the liquid-solid pair

The equilibrium force equation (along the solid surface) is

\[ T_3 \cos \theta + T_1 - T_2 = 0 \]

\[ \therefore \ \cos \theta = \frac{T_2 - T_1}{T_3} \] … (1)

Case (1) : If \( T_2 > T_1 \), \( \cos \theta \) is positive and contact angle \( \theta < 90^\circ \), so that the liquid wets the surface.

Case (2) : If \( T_2 < T_1 \), \( \cos \theta \) is negative and \( \theta \) is obtuse, so that the liquid is non-wetting.

Case (3) : If \( T_2 - T_1 \approx T_3 \), \( \cos \theta = 1 \) and \( \theta \approx 0^\circ \).

Case (4) : If \( T_2 - T_1 > T_3 \), \( \cos \theta \) will be greater than 1 which is impossible, so that there will be no equilibrium and the liquid will spread over the solid surface.

Q. 60. State the expression for the angle of contact in terms of interfacial tensions? (1 mark)

Ans. \( \cos \theta = \frac{T_2 - T_1}{T_3} \), where \( \theta \) is the angle of contact for a liquid–solid pair, \( T_1 \) is the liquid-solid interfacial tension, \( T_2 \) is the solid-gas (air + vapour) interfacial tension and \( T_3 \) is the liquid-gas interfacial tension.

Q. 61. In terms of interfacial tension, when is the angle of contact acute? (1 mark)

Ans. The angle of contact is acute when the solid-gas (air + vapour) interfacial tension is greater than the liquid-solid interfacial tension.

Q. 62. In terms of interfacial tensions, when is the angle of contact obtuse? (1 mark)

Ans. The angle of contact is obtuse when the solid-gas (air + vapour) interfacial tension is less than the liquid-solid interfacial tension.

Q. 63. Explain: Pure water on a clean glass surface tends to spread out, while mercury on the same surface tends to form a drop. (3 marks)

Ans. Refer to the answer to Q. 58 up to Eq. (1) and continue:

For pure water on a clean glass surface, \( T_2 - T_1 \approx T_3 \), so that \( \cos \theta \approx 1 \) and \( \theta \approx 0^\circ \), i.e., pure water completely wets the surface. Thus, when a drop of pure water is put on a clean glass surface, it flattens out. For mercury on the same surface, \( T_2 < T_1 \), so that \( \cos \theta \) is negative and \( \theta \) is obtuse, i.e., mercury is non-wetting. Thus, a small drop of mercury on a clean glass surface is almost spherical.

Q. 64. State the factors affecting a liquid-solid angle of contact. (\( \frac{1}{2} \) mark each)

Ans. Factors affecting a liquid-solid angle of contact:

(1) the nature of the liquid and the solid in contact,
(2) impurities in the liquid,
(3) temperature of the liquid.

Q. 65. Explain the effect of impurity on the angle of contact (or surface tension of a liquid). (2 marks)

Ans. Effect of impurity:

(i) The angle of contact or the surface tension of a liquid increases with dissolved impurities like com-
mon salt. For dissolved impurities, the angle of contact (or surface tension) increases linearly with the concentration of the dissolved materials.

(ii) It decreases with sparingly soluble substances like phenol or alcohol. A detergent is a surfactant whose molecules have hydrophobic and hydrophilic ends; the hydrophobic ends decrease the surface tension of water. With reduced surface tension, the water can penetrate deep into the fibres of a cloth and remove stubborn stains.

(iii) It decreases with insoluble surface impurities like oil, grease or dust. For example, mercury surface contaminated with dust does not form perfect spherical droplets till the dust is removed.

Q. 66. Explain the effect of temperature on the angle of contact (or surface tension of a liquid). (2 marks)

Ans. Effect of temperature: The surface tension of a liquid decreases with increasing temperature of the liquid. For small temperature differences, the decrease in surface tension is nearly directly proportional to the temperature rise.

If \( T \) and \( T_0 \) are the surface tensions of a liquid at temperatures \( \theta \) and \( 0 \) \( ^\circ \)C, respectively, then \( T = T_0 (1 - \alpha \theta) \) where \( \alpha \) is a constant for a given liquid. The surface tension of a liquid becomes zero at its critical temperature. The surface tension increases with increasing temperature only in case of molten copper and molten cadmium.

Q. 67. Why cold wash is recommended for new cotton fabrics while hot wash for removing stains? (1 mark)

Ans. Cold wash is recommended for new/coloured cotton fabrics. Cold water, due to its higher surface tension, does not penetrate deep into the fibres and thus does not fade the colours. Hot water, because of its lower surface tension, can penetrate deep into fabric fibres and remove tough stains.

Q. 68. Explain in brief the pressure difference across a curved liquid surface. (4 marks)

Ans. Every molecule lying within the surface film of a static liquid is pulled tangentially by forces due to surface tension. The direction of their resultant, \( \vec{F}_T \) on a molecule depends upon the shape of that liquid surface and decides the cohesion pressure at a point just below the liquid surface.

Consider two molecules, A and B, respectively just above and below the free surface of a liquid. So, the level difference between them is negligibly small and the atmospheric pressure on both is the same, \( p_0 \). Let \( \vec{F}_{atm} \) be the downward force on A and B due to the atmospheric pressure.

If the free surface of a liquid is horizontal, the resultant force \( \vec{F}_T \) on molecule B is zero, Fig. 2.20 (a). Then, the cohesion pressure is negligible and the net force on A and B is \( \vec{F}_{atm} \). The pressure difference on the two sides of the liquid surface is zero.

If the free surface of a liquid is concave, the resultant force \( \vec{F}_T \) on molecule B is outwards (away from the liquid), Fig. 2.20 (b), opposite to \( \vec{F}_{atm} \). Then, the net force on B is less than \( \vec{F}_{atm} \) and the cohesion pressure is decreased. The pressure above the concave liquid surface is greater than that just below the liquid surface.

If the free surface of a liquid is convex, the resultant force \( \vec{F}_T \) on molecule B acts inwards (into the liquid), Fig., 2.20 (c), in the direction of \( \vec{F}_{atm} \). Then, the net force on B is greater than \( \vec{F}_{atm} \) and the cohesion pressure is increased. The pressure below the convex liquid surface is greater than that just above the liquid surface.

Q. 69. Derive an expression for the excess pressure inside a liquid drop.

OR

Derive Laplace’s law for a spherical membrane.

Ans. Consider a small spherical liquid drop with a radius \( R \). It has a convex surface, so that the pressure \( p \) on the concave side (inside the liquid) is
greater than the pressure \( p_0 \) on the convex side (outside the liquid). The surface area of the drop is
\[
A = 4\pi R^2 \quad \text{(1)}
\]
Imagine an increase in radius by an infinitesimal amount \( dR \) from the equilibrium value \( R \). Then, the differential increase in surface area would be
\[
dA = 8\pi R \cdot dR \quad \text{(2)}
\]
The increase in surface energy would be equal to the work required to increase the surface area:
\[
dW = T \cdot dA = 8\pi TRdR \quad \text{(3)}
\]
Derive Laplace’s law for spherical membrane of a bubble due to surface tension. \( (3 \text{ marks}) \)

**Ans.** Consider a small, spherical, thin-filmed soap bubble with a radius \( R \). Let the pressure outside the drop be \( p_0 \) and that inside be \( p \). A soap bubble in air is like a spherical shell and has two gas-liquid interfaces. Hence, the surface area of the bubble is
\[
A = 8\pi R^2 \quad \text{(1)}
\]
Hence, with a hypothetical increase in radius by an infinitesimal amount \( dR \), the differential increase in surface area and surface energy would be
\[
dA = 16\pi R \cdot dR \quad \text{and}
\[dW = T \cdot dA = 16\pi TRdR \quad \text{(2)}
\]
We assume that \( dR \) is so small that the pressure inside remains the same, equal to \( p \). All parts of the surface of the bubble experiences an outward force per unit area equal to \( p - p_0 \). Therefore, the work done by this outward pressure-developed force against the surface tension force during the increase in radius \( dR \) is
\[
dW = (\text{excess pressure} \times \text{surface area}) \cdot dR = (p - p_0) \times 4\pi R^2 \cdot dR \quad \text{(3)}
\]
From Eqs. (2) and (3),
\[
(p - p_0) \times 4\pi R^2 \cdot dR = 8\pi TRdR
\]
\[
\therefore \quad p - p_0 = \frac{2T}{R} \quad \text{(5)}
\]
which is called Laplace’s law for a spherical membrane (or Young-Laplace equation in spherical form).

**Notes:** (1) The above method is called the principle of virtual work. (2) Equation (5) also applies to a gas bubble within a liquid, and the excess pressure in this case is also called the gauge pressure. An air or gas bubble within a liquid is technically called a cavity because it has only one gas-liquid interface. A bubble, on the other hand, such as a soap bubble, has two gas-liquid interfaces.

**Q. 70.** Derive an expression for the excess pressure inside a soap bubble.

**OR**

**Q. 71.** What is the excess pressure inside a soap bubble of radius 3 cm if the surface tension of the soap solution is 30 dyn/cm? \( (1 \text{ mark}) \)

**Ans.** Excess pressure, \( p - p_0 = \frac{4T}{R} = \frac{4 \times 30}{3} = 40 \text{ dyn/cm}^2 \)

**Q. 72.** Two soap bubbles of the same soap solution have radii 3 cm and 1.5 cm. If the excess pressure inside the bigger bubble is 40 dyn/cm\(^2\), what is the excess pressure inside the smaller bubble? \( (1 \text{ mark}) \)

**Ans.** Excess pressure \( \propto T/R \). In this case, the surface tension is the same in the two cases. Hence, the
excess pressure inside the smaller bubble will be 80 dyn/cm².

Q. 73. Explain : In the absence of gravity or other external forces, a liquid drop assumes a spherical shape. 

(2 marks)

Ans. A spherical shape has the minimum surface area-to-volume ratio of all geometric forms. If any external force distorts the sphere, molecules must be brought from the interior to the surface in order to provide for the increased surface area. This process requires work to be done in order to raise the potential energy of a molecule. The change in free surface energy is equal to the net work done to alter the surface area of the liquid.

However, spontaneous processes are associated with a decrease in free energy. Hence, in the absence of external forces, a liquid drop will spontaneously assume a spherical shape in order to minimize its exposed surface area and thereby its free surface energy.

[Note : The spontaneous coalescence of two similar liquid droplets into one large drop when brought into contact is a dramatic demonstration of the decrease in free surface energy brought about by the decrease in total surface area by the formation of a single larger drop.]

Q. 74. A small air bubble of radius r in water is at a depth h below the water surface. If p₀ is the atmospheric pressure, ρ is the density of water and T is the surface tension of water, what is the pressure inside the bubble? 

(1 mark)

Ans. The absolute pressure within the liquid at a depth h is \( p = p₀ + ρgh \).

Since the excess pressure inside a bubble is \( \frac{2T}{R} \),

the pressure inside the bubble is

\[ p_m = p + \frac{2T}{R} = p₀ + ρgh + \frac{2T}{R}. \]

Brain teaser

(Textbook page 41)

1. Can you suggest any method to measure the surface tension of a soap solution? Will this method have any commercial application?

There are more than 40 methods for determining equilibrium surface tension at the liquid-fluid and solid-fluid boundaries. Measuring the capillary rise (see Unit 2.4.7) is the laboratory method to determine surface tension.

Among the various techniques, equilibrium surface tension is most frequently measured with force tensiometers or optical (or the drop profile analysis) tensiometers in customized measurement setups.

[See https://www.biolinscientific.com/measurements/surface-tension]

2. What happens to surface tension under different gravity (e.g., aboard the International Space Station or on the lunar surface)?

Surface tension does not depend on gravity.

[Note : The behaviour of liquids on board an orbiting spacecraft is mainly driven by surface tension phenomena. These make predicting their behaviour more difficult than under normal gravity conditions (i.e., on the Earth’s surface). New challenges appear when handling liquids on board a spacecraft, which are not usually present in terrestrial environments. The reason is that under the weightlessness (or almost weightlessness) conditions in an orbiting spacecraft, the different inertial forces acting on the bulk of the liquid are almost zero, causing the surface tension forces to be the dominant ones. In this ‘micro-gravity’ environment, the surface tension forms liquid drops into spheres to minimize surface area, causes liquid columns in a capillary rise up to its rim (without over-flowing). Also, when a liquid drop impacts on a dry smooth surface on the Earth, a splash can be observed as the drop disintegrates into thousands of droplets. But no splash is observed as the drop hits dry smooth surface on the Moon. The difference is the atmosphere. As the Moon has no atmosphere, and therefore no gas surrounding a falling drop, the drop on the Moon does not splash.

(See http://mafija.fmf.uni-lj.si/]

Solved Problems 2.4.3 – 2.4.6

Q. 75. Solve the following :

(1) What is the excess pressure (in atm) inside a soap bubble with a radius of 1.5 cm and surface tension of \( 3 \times 10^{-2} \) N/m? [1 atm = 101.3 kPa]  

(2 marks)

Solution :

Data : \( R = 1.5 \times 10^{-2} \) m, \( T = 3 \times 10^{-2} \) N/m, 

1 atm = 1.013 \times 10^5 Pa
The excess pressure inside a soap bubble is

\[ p - p_0 = \frac{4T}{R} \]

\[ = \frac{4 \times 3 \times 10^{-2}}{1.5 \times 10^{-2}} = 8 \text{ Pa} \]

\[ = \frac{8}{1.013 \times 10^5} \text{ atm} = 7.897 \times 10^{-5} \text{ atm} \]

(2) A raindrop of diameter 4 mm is about to fall on the ground. Calculate the pressure inside the raindrop. [Surface tension of water \( T = 0.072 \text{ N/m} \), atmospheric pressure = \( 1.013 \times 10^5 \text{ N/m}^2 \)]

Solution :

Data: \( D = 4 \times 10^{-3} \text{ m}, \ T = 0.072 \text{ N/m}, \ p_0 = 1.013 \times 10^5 \text{ N/m}^2 \)

\[ R = \frac{D}{2} = 2 \times 10^{-3} \text{ m} \]

The excess pressure inside the raindrop is

\[ p - p_0 = \frac{2T}{R} \left( \frac{2(0.072)}{2 \times 10^{-3}} \right) = 72 \text{ N/m}^2 \]

\[ \therefore p = 101300 + 72 = 101372 \text{ N/m}^2 \]

(3) What should be the diameter of a soap bubble such that the excess pressure inside it is 51.2 Pa? [Surface tension of soap solution = \( 3.2 \times 10^{-2} \text{ N/m} \)]

Solution :

Data: \( p - p_0 = 51.2 \text{ Pa}, \ T = 3.2 \times 10^{-2} \text{ N/m} \)

For a soap bubble, \( p - p_0 = \frac{4T}{R} \)

\[ \therefore \text{ The radius of the soap bubble should be} \]

\[ R = \frac{4T}{p - p_0} = \frac{4 \times 3.2 \times 10^{-2}}{51.2} = 2.5 \times 10^{-3} \text{ m} = 2.5 \text{ mm} \]

\[ \therefore \text{ the diameter of the soap bubble should be} \]

\[ 2 \times 2.5 = 5 \text{ mm}. \]

(4) The lower end of a capillary tube of diameter 1 mm is dipped 10 cm below the water surface in a beaker. What pressure is required to blow a hemispherical air bubble at the lower end of the tube? Present your answer rounded off to 4 significant figures. [Surface tension = 0.072 N/m, density = \( 10^3 \text{ kg/m}^3 \), atmospheric pressure = 101.3 kPa, \( g = 9.8 \text{ m/s}^2 \)]

Solution :

Data: \( R = 10^{-3} \text{ m}, \ T = 0.072 \text{ N/m}, \ p = 10^3 \text{ kg/m}^3, \ h = 0.1 \text{ m} \)

Let the atmospheric pressure be \( p_0 \). Then, the absolute pressure within the liquid at a depth \( h \) is

\[ p = p_0 + \rho gh \]

Hence, the pressure inside the bubble is
Q. 75. State any four applications of capillarity. (2 marks)

Ans. Applications of capillarity:

1. A blotting paper or a cotton cloth absorbs water, ink by capillary action.
2. Oil rises up the wick of an oil lamp and sap rises up xylem tissues of a tree by capillarity.
3. Ground water rises to the open surface through the capillaries formed in the soil. In summer, the farmers plough their fields to break these capillaries and prevent excessive evaporation.
4. Water rises up the crevices in rocks by capillary action. Expansion and contraction of this water due to daily and seasonal temperature variations cause the rocks to crumble.

[Note: The rise of sap is due to the combined action of capillarity and transpiration. The transpiration pull is considered to be the major driving force for water transport throughout a plant.]

Q. 76. What is capillary? What is capillarity or capillary action? (2 marks)

Ans. (1) A tube of narrow bore (i.e. very small diameter) is called a capillary tube. The word capillary is derived from the Latin capillus meaning hair, capillaris in Latin means ‘like a hair’.

(2) If a capillary tube is just partially immersed in a non-wetting liquid, the liquid rises in the capillary tube. This is called capillary rise.

If a capillary tube is just partially immersed in a non-wetting liquid, the liquid falls in the capillary tube. This is called capillary depression.

The rise of a wetting liquid and fall of a non-wetting liquid in a capillary tube is called capillarity.

Q. 77. Explain the phenomenon of capillarity. (4 marks)

OR

Explain the capillary action. (4 marks)

Explain (1) the rise of a liquid (2) the fall of a liquid in a capillary on the basis of pressure difference. (2 marks)

Ans. (1) When a capillary tube is partially immersed in a wetting liquid, there is capillary rise and the liquid meniscus inside the tube is concave, as shown in Fig. 2.22 (a).

Consider four points A, B, C, D, of which point A is just above the concave meniscus inside the capillary and point B is just below it. Points C and D are just above and below the free liquid surface outside.

Let $P_A$, $P_B$, $P_C$ and $P_D$ be the pressures at points A, B, C and D, respectively.

Now, $P_A = P_C =$ atmospheric pressure

The pressure is the same on both sides of the free surface of a liquid, so that
Fig. 2.22: Explanation of (a) capillary rise
(b) capillary depression

\[ P_C = P_D \]
\[ \therefore P_A = P_D \]

The pressure on the concave side of a meniscus is always greater than that on the convex side, so that

\[ P_A > P_B \]
\[ \therefore P_D > P_B \]  \( (\therefore P_A = P_D) \)

The excess pressure outside presses the liquid up the capillary until the pressures at B and D (at the same horizontal level) equalize, i.e., \( P_B \) becomes equal to \( P_D \). Thus, there is a capillary rise.

(2) For a non-wetting liquid, there is capillary depression and the liquid meniscus in the capillary tube is convex, as shown in Fig. 2.22(b).

Consider again four points A, B, C and D when the meniscus in the capillary tube is at the same level as the free surface of the liquid. Points A and B are just above and below the convex meniscus. Points C and D are just above and below the free liquid surface outside.

The pressure at B \( (P_B) \) is greater than that at A \( (P_A) \). The pressure at A is the atmospheric pressure \( H \) and at D, \( P_D \approx H = P_A \). Hence, the hydrostatic pressure at the same levels at B and D are not equal, \( P_B > P_D \). Hence, the liquid flows from B to D and the level of the liquid in the capillary falls. This continues till the pressure at B’ is the same as that D’, that is till the pressures at the same level are equal.

Q. 79. Assuming Laplace’s law for a spherical membrane, derive the expression for the capillary rise of a wetting liquid.

Obtain the relation between surface tension and rise of a liquid in a capillary tube using Laplace’s formula for a spherical membrane.

★ Derive an expression for capillary rise for a liquid having a concave meniscus.  \hspace{1cm} (3 marks)

Ans. Consider a capillary tube of radius \( r \) partially immersed into a wetting liquid of density \( \rho \). Let the capillary rise be \( h \) and \( \theta \) be the angle of contact at the edge of contact of the concave meniscus and glass (Fig. 2.23). If \( R \) is the radius of curvature of the meniscus then from the figure, \( r = R \cos \theta \).

Surface tension \( T \) is the tangential force per unit length acting along the contact line. It is directed into the liquid making an angle \( \theta \) with the capillary wall. We ignore the small volume of the liquid in the meniscus. The gauge pressure within the liquid at a depth \( h \), i.e., at the level of the free liquid surface open to the atmosphere, is

\[ p - p_0 = \rho gh \]  \( \ldots \) (1)

By Laplace’s law for a spherical membrane, this gauge pressure is

\[ p - p_0 = \frac{2T}{R} \]  \( \ldots \) (2)

\[ \therefore h\rho g = \frac{2T}{R} \approx \frac{2T \cos \theta}{r} \]
\[ \therefore h = \frac{2T \cos \theta}{\rho g} \]  \( \ldots \) (3)
Thus, narrower the capillary tube, the greater is the capillary rise.

From Eq. (3),

\[ T = \frac{hr_{\mu g}}{2T \cos \theta} \]  \hspace{1cm} \ldots (4)

Equations (3) and (4) are also valid for capillary depression \( h \) of a non-wetting liquid. In this case, the meniscus is convex and \( \theta \) is obtuse. Then, cos \( \theta \) is negative but so is \( h \), indicating a fall or depression of the liquid in the capillary. \( T \) is positive in both cases.

[Note: The capillary rise \( h \) is called Jurin height, after James Jurin who studied the effect in 1718. For capillary rise, Eq. (3) is also called the ascent formula.]

Q. 80. Derive the expression for the capillary rise of a wetting liquid using forces on the liquid column.

OR

Obtain the relation between surface tension and rise of a liquid in a capillary tube using forces on the liquid column.

★ Derive an expression for capillary rise for a liquid having a concave meniscus.  \hspace{1cm} (3 marks)

Ans. Consider a capillary tube of radius \( r \) partially immersed into a wetting liquid of density \( \rho \). Let the capillary rise be \( h \) and \( \theta \) be the angle of contact at the edge of contact of the concave meniscus and glass (Fig. 2.24)

\[ T = \frac{hr_{\mu g} \cos \theta}{2T} \]  \hspace{1cm} \ldots (1)

\[ V = \pi r^2 \text{ constant} \] and angle of contact \( \theta \).

Equation (4) is also valid for capillary depression \( h \) of a non-wetting liquid.

[Note: If the small volume of the liquid in the meniscus is also taken into account, then \( h \) must be replaced by \( \frac{h + r}{3} \), so that the above formula becomes

\[ T = \frac{hr_{\mu g} \cos \theta}{2T} \left( \frac{h + \frac{r}{3}}{3} \right) \]

which reduces to Eq. (4) for \( r \ll h \).]

Q. 81. Draw a neat labelled diagram showing the forces acting on the meniscus of water in a capillary tube.  \hspace{1cm} (2 marks)

Ans. Refer Fig. 2.24.

Q. 82. Two capillary tubes have radii in the ratio 1 : 2.

If they are dipped in the same liquid, what will be the ratio of capillary rise in the two tubes?

(1 mark)

Ans. \( T = \frac{hr_{\mu g}}{2 \cos \theta} \). In this case, \( hr \) is constant

\[ \therefore h_1 : h_2 = r_2 : r_1 = 2 : 1 \]

Q. 83. The radii of two columns of a U-tube are \( r_1 \) and \( r_2 \). When a liquid of density \( \rho \) and angle of contact \( \theta = 0^\circ \) is filled in it, the level difference of the liquid in the two columns is \( h \). Find the surface tension of the liquid.  \hspace{1cm} (2 marks)
Ans. Capillary rise, \( h = \frac{2T \cos \theta}{\rho g} \), where \( \theta \) is the angle of contact.

Assuming the two columns of the U-tube to be sufficiently thin,

\[
h_1 = \frac{2T}{r_1 \rho g} \quad \text{and} \quad h_2 = \frac{2T}{r_2 \rho g}
\]

since \( \theta = 0^\circ \) (given)

If \( r_1 < r_2 \), \( h_1 > h_2 \), so that

the level difference of the liquid in the two columns,

\[
h = h_1 - h_2 = \frac{2T}{r_1 \rho g} - \frac{2T}{r_2 \rho g} = \frac{2}{\rho g} \left( \frac{1}{r_1} - \frac{1}{r_2} \right)
\]

\[
= \frac{2T}{\rho g} \left( \frac{r_2 - r_1}{r_1 r_2} \right)
\]

\[\therefore\] The surface tension of the liquid, \( T = \frac{h \rho g r_2}{2(r_2 - r_1)} \).

Do you know?

To read height of a liquid in a capillary, we read the position of the tangent to the liquid meniscus, concave or convex. The observer’s eye should be as shown.

Solved Problems 2.4.7

Q. 84. Solve the following:

(1) A liquid of density 900 kg/m\(^3\) rises to a height of 9 mm in a capillary tube of 2.4 mm diameter. If the angle of contact is 25°, find the surface tension of the liquid. (2 marks)

Solution:

Data: \( \rho = 900 \text{ kg/m}^3 \), \( h = 9 \text{ mm} = 9 \times 10^{-3} \text{ m} \), \( \theta = 25^\circ \), \( g = 9.8 \text{ m/s}^2 \)

\[
r = \frac{1}{2} \times \text{diameter} = \frac{2.4}{2} = 1.2 \text{ mm} = 1.2 \times 10^{-3} \text{ m}
\]

\[\cos \theta = \cos 25^\circ = 0.9063\]

The surface tension of the liquid,

\[
T = \frac{rh \rho g}{2 \cos \theta} = \frac{1.2 \times 10^{-3} \times 9 \times 10^{-3} \times 900 \times 9.8}{2 \times 0.9063} = 5.257 \times 10^{-2} \text{ N/m}
\]

(2) A capillary tube of uniform bore is dipped vertically in water which rises by 7 cm in the tube. Find the radius of the capillary tube if the surface tension of water is 70 dyn/cm. \( [g = 980 \text{ cm/s}^2] \) (2 marks)

Solution:

Data: \( h = 7 \text{ cm}, T = 70 \text{ dyn/cm}, \rho = 1 \text{ g/cm}^3 \) and \( \theta = 0^\circ \) (for water)

\[\therefore\] \( \cos \theta = 1 \)

Surface tension, \( T = \frac{rh \rho g}{2 \cos \theta} \)

\[\therefore\] The radius of the capillary tube,

\[
r = \frac{2T \cos \theta}{h \rho g} = \frac{2 \times 70 \times 1}{7 \times 1 \times 980} = 0.02041 \text{ cm}
\]

★ (3) Calculate the rise of water in a clean glass capillary tube of radius 0.1 mm when dipped into water of surface tension \( 7 \times 10^{-2} \text{ N/m} \). [Angle of contact between water and glass = 0, density of water = 1000 kg/m\(^3\), \( g = 9.8 \text{ m/s}^2 \)] (2 marks)

Solution:

Data: \( r = 0.1 \text{ mm} = 1 \times 10^{-4} \text{ m}, \theta = 0^\circ \),
\( T = 7 \times 10^{-2} \text{ N/m}, r = 10^3 \text{ kg/m}^3, g = 9.8 \text{ m/s}^2 \)

\[\cos \theta = \cos 0^\circ = 1\]

Capillary rise, \( h = \frac{2T \cos \theta}{r \rho g} = \frac{2 \times 7 \times 10^{-2} \times 1}{1 \times 10^{-3} \times 10^3 \times 9.8} = 0.143 \text{ m} \)

(4) A liquid rises to a height of 9 cm in a glass capillary tube of radius 0.02 cm. What will be the height of the liquid column in a glass capillary tube of radius 0.03 cm? (2 marks)

Solution:

Data: \( h_1 = 9 \text{ cm}, r_1 = 0.02 \text{ cm}, r_2 = 0.03 \text{ cm} \)

For the first capillary, \( T = \frac{r_1 h_1 \rho g}{2 \cos \theta} \)

For the second capillary, \( T = \frac{r_2 h_2 \rho g}{2 \cos \theta} \)

\[\therefore\] \( \frac{r_1 h_1 \rho g}{2 \cos \theta} = \frac{r_2 h_2 \rho g}{2 \cos \theta} \)

\[\therefore\] \( r_1 h_1 = r_2 h_2 \)
(5) Water rises to a height of 5 cm in a certain capillary tube. In the same capillary tube, mercury is depressed by 2.02 cm. Compare the surface tensions of water and mercury.

[Density of water = 1000 kg/m³, density of mercury = 13600 kg/m³, angle of contact for water = 0°, angle of contact for mercury = 148°]

Solution: Let $T_w$, $\theta_w$, $h_w$ and $\rho_w$ be the surface tension, angle of contact, capillary rise and density of water respectively. Let $T_m$, $\theta_m$, $h_m$ and $\rho_m$ be the corresponding quantities for mercury. The radius ($r$) of the capillary is the same in both cases.

Data: $h_w = 5 \text{ cm} = 5 \times 10^{-2} \text{ m}$, $\theta_w = 0^\circ$,

\[ \rho_w = 1000 \text{ kg/m}^3, \quad h_m = -2.02 \text{ cm} \]
\[ = -2.02 \times 10^{-2} \text{ m}, \]
\[ \rho_m = 13600 \text{ kg/m}^3, \quad \theta_m = 148^\circ \]

[Note: $h_m$ is taken to be negative because for mercury there is capillary depression.]

\[ \cos \theta_w = \cos 0^\circ = 1 \]
\[ \cos \theta_m = \cos 148^\circ = -\cos 32^\circ = -0.8480 \]

\[ T_w = \frac{h_w \rho_w g}{2 \cos \theta_w} \text{ and } T_m = \frac{h_m \rho_m g}{2 \cos \theta_m} \]

\[ \frac{T_w}{T_m} = \frac{h_w \rho_w \cos \theta_m}{h_m \rho_m \cos \theta_w} \]

\[ \frac{T_w}{T_m} = \frac{5 \times 10^{-2} \times 1000 \times (-0.8480)}{-2.02 \times 10^{-2} \times 13600 \times 1} = 0.1543 \]

(6) When a glass capillary tube of radius 0.4 mm is dipped into mercury, the level of mercury inside the capillary stands 1.50 cm lower than that outside. Calculate the surface tension of mercury.

[Angle of contact of mercury with glass = 148°, density of mercury = 13600 kg/m³]

Solution:

Data: $r = 0.4 \text{ mm} = 4 \times 10^{-4} \text{ m}$,

\[ h = -1.50 \text{ cm} = -1.50 \times 10^{-2} \text{ m}, \]
\[ \rho = 13.6 \times 10^3 \text{ kg/m}^3, \quad g = 9.8 \text{ m/s}^2, \quad \theta = 148^\circ \]

\[ \cos \theta = \cos 148^\circ = -\cos 32^\circ = -0.8480 \]

The surface tension of mercury is

\[ T = \frac{rh \rho g}{2 \cos \theta} \]
\[ = \frac{(4 \times 10^{-4})(-1.50 \times 10^{-2})(13.6 \times 10^3)(9.8)}{2(-0.8480)} \]
\[ = 0.4715 \text{ N/m} \]

(7) The tube of a mercury barometer is 1 cm in diameter. What correction due to capillarity is to be applied to the barometric reading if the surface tension of mercury is 435.5 dyn/cm and the angle of contact of mercury with glass is 140°?

[Density of mercury = 13600 kg/m³]

Solution:

Data: $d = 1 \text{ cm}$, $T = 435.5 \text{ dyn/cm}$, $\theta = 140^\circ$,

\[ \rho = 13660 \text{ kg/m}^3, \quad g = 9.8 \text{ m/s}^2, \quad r = \frac{d}{2} = 0.5 \text{ cm}, \quad \cos 140^\circ = -0.7660 \]

\[ T = \frac{hr \rho g}{2 \cos \theta} \quad \therefore \quad h = 2 \frac{T \cos \theta}{r \rho g} \]
\[ = \frac{2 \times 435.5 \times (-0.7660)}{0.5 \times 13.6 \times 980} = -0.1001 \text{ cm} \]

\[ \therefore \quad \text{The correction due to capillarity} = +0.1001 \text{ cm} \]

(8) Calculate the density of paraffin oil, if within a glass capillary of diameter 0.25 mm dipped in paraffin oil of surface tension 0.0245 N/m, the oil rises to a height of 4 cm. [Angle of contact of paraffin oil with glass = 28°, acceleration due to gravity = 9.8 m/s²]

Solution:

Data: $d = 0.25 \text{ mm}$, $T = 0.0245 \text{ N/m}$,

\[ h = 4 \text{ cm} = 4 \times 10^{-2} \text{ m}, \quad \theta = 28^\circ, \quad g = 9.8 \text{ m/s}^2 \]

\[ \therefore \quad r = \frac{d}{2} = 0.125 \text{ mm} = 1.25 \times 10^{-4} \text{ m} \]

\[ T = \frac{hr \rho g}{2 \cos \theta} \]
\[ = \frac{2T \cos \theta}{hr g} \]
\[ = \frac{2 \times 0.0245 \times \cos 28^\circ}{4 \times 10^{-2} \times 1.25 \times 10^{-4} \times 9.8} \]
\[ = \frac{0.0245 \times 0.8829}{2.5 \times 9.8 \times 10^{-6}} = 882.9 \text{ kg/m}^3 \]

This gives the density of paraffin oil.
(9) A capillary tube of radius \( r \) can support a liquid column of weight \( 6.284 \times 10^{-4} \) N. Calculate the radius of the capillary if the surface tension of the liquid is \( 4 \times 10^{-2} \) N/m. \( (2 \text{ marks}) \)

Solution :

Data : \( mg = 6.284 \times 10^{-4} \) N, \( T = 4 \times 10^{-2} \) N/m

Net upward force = weight of liquid column
\( \therefore 2\pi r T \cos \theta = mg \)

Assuming the angle of contact, \( \theta = 0^\circ \) (\( \therefore \) data not given), the radius of the capillary is

\[
 r = \frac{mg}{2\pi T} = \frac{6.284 \times 10^{-4}}{2(3.142)(4 \times 10^{-2})} = 2.5 \times 10^{-3} \text{ m}
\]

(10) Two vertical glass plates are held parallel 0.5 mm apart, dipped in water. If the surface tension of water is 70 dyn/cm, calculate the height to which water rises between the two plates. \( (2 \text{ marks}) \)

Solution :

Data : \( x = 0.5 \text{ mm} = 5 \times 10^{-4} \text{ m}, T = 0.07 \text{ N/m}, \rho (\text{water}) = 10^3 \text{ kg/m}^3 \)

Let the width of each glass plate be \( b \) and the height to which the water rises between the plates be \( h \).

Then, the upward force on the water between the plates due to surface tension = \( 2Tb \cos \theta \)

where \( \theta \) is the angle of contact of water with glass.

The weight of the water between the plates = \( (bxh\rho)g = g \)

where \( x \) is the separation between the plates and \( \rho \) is the density of water.

Equating, \( (bxh\rho)g = 2Tb \cos \theta \)

\( \therefore \) The height to which water rises between the two plates,

\[
 h = \frac{2T \cos \theta}{x \rho g} = \frac{2(7 \times 10^{-2})(1)}{5 \times 10^{-4}(10^3)(9.8)} = \frac{0.2}{7} = 0.02857 \text{ m} = 2.857 \text{ cm}
\]

(11) A glass capillary of radius 0.4 mm is inclined at 60° with the vertical in water. Find the length of water column in the capillary tube. \( \text{[Surface tension of water} = 7 \times 10^{-2} \text{ N/m]} \)

Solution :

Data : \( r = 4 \times 10^{-4} \text{ m}, \phi = 60^\circ, T = 7 \times 10^{-2} \text{ N/m} \)

Let \( h \) be the capillary rise when the capillary tube is immersed vertically in water. Let \( l \) be the length of the water column in the capillary tube above that of the outside level.

\[
 h = \frac{2T \cos \theta}{r \rho g} \quad \text{and} \quad l \cos \phi = h
\]

where \( \theta \) is the angle of contact and \( \phi \) is the angle of inclination of the capillary tube with the vertical.

\[
 l = \frac{2T \cos \theta}{r \rho g \cos \phi} = \frac{2(7 \times 10^{-2}) \cos 0^\circ}{(4 \times 10^{-4})(10^3)(9.8) \cos 60^\circ}
\]

\[
 = \frac{0.2}{2 \times 9.8 \times 0.5 \times \frac{1}{2}} = \frac{1}{14} = 0.07143 \text{ m} = 7.143 \text{ cm}
\]

Units

2.5 Fluids in motion
2.6 Critical velocity and Reynolds number
2.6.1 Viscosity
2.6.2 Coefficient of viscosity
2.7 Stokes’ Law
2.7.1 Terminal velocity

Q. 85. What is hydrodynamics? \( (1 \text{ mark}) \)

Ans. Hydrodynamics is the branch of physics that deals with fluid dynamics, i.e., the study of fluids in motion. Since the most basic fluid motion is highly complex, we consider only ideal fluids-non-viscous and incompressible, i.e., fluids whose internal friction is negligible and density is constant throughout.
Q. 86. What is meant by a steady flow?  (1 mark)
Ans. When a liquid flows slowly over a surface or through a pipe such that its velocity or pressure at any point within the fluid is constant, it is said to be in steady flow.

Q. 87. Explain a streamline and streamline flow.  
(2 marks)
Ans. Streamline: Consider point A, Fig. 2.27, within a fluid. The velocity \( \vec{v} \) at A does not change with time. Hence, every particle passes point A with the same speed and in the same direction. The same is true about the other points such as B and C. A curve which is tangent or parallel to the velocity of the fluid particles at every point will be the path of every particle arriving at A. It is called a streamline. A fluid particle cannot cross a streamline but only flow along it.

Streamline flow: When a liquid flows slowly over a surface or through a pipe with a velocity less than a certain critical velocity, the motion of its molecules is orderly. All molecules passing a given point proceed with the same velocity. This kind of fluid motion is called streamline or steady flow.

Q. 88. Explain a flow tube.  (2 marks)
Ans. A bundle of adjacent streamlines form a tube of flow or flow tube through which the fluid is flowing. In a flow tube, where the streamlines are close together the velocity is high, and where they are widely separated, the fluid is moving slowly. No fluid can cross the boundary of a tube of flow.

Q. 89. Explain turbulent flow.  (2 marks)
Ans. Turbulent flow or turbulence is a non-steady fluid flow in which streamlines and flowtubes change continuously. It has two main causes. First, any obstruction or sharp edge, such as in a tap, creates turbulence by imparting velocities perpendicular to the flow. Second, if the speed with which a fluid moves relative to a solid body is increased beyond a certain critical velocity the flow becomes unstable or one of extreme disorder. In both cases, the fluid particles still move in general towards the main direction as before. But now all sorts of secondary motions cause them to cross and recross the main direction continuously. The orderly streamlines break up into eddies or vortices and the result is turbulence. In a turbulent flow, regions of fluid move in irregular, colliding paths, resulting in mixing and swirling (Fig. 2.30).
What would happen if two streamlines intersect?

The velocity of a fluid molecule is always tangential to the streamline. If two streamlines intersect, the velocity at that point will not be constant or unique.

Identify some examples of streamline flow and turbulent flow in everyday life. How would you explain them? When would you prefer a streamline flow?

Smoke rising from an incense stick inside a windless room, air flow around a car or aeroplane in motion are some examples of streamline flow. Fish, dolphins, and even massive whales are streamlined in shape to reduce drag. Migratory birds species that fly long distances often have particular features such as long necks, and flocks of birds fly in the shape of a spearhead as that forms a streamlined pattern.

Turbulence results in wasted energy. Cars and aeroplanes are painstakingly streamlined to reduce fluid friction, and thus the fuel consumption. Turbulence is commonly seen in washing machines and kitchen mixers. Turbulence in these devices is desirable because it causes mixing. Recent developments in high-speed videography and computational tools for modelling is rapidly advancing our understanding of the aerodynamics of bird and insect flights which fascinate both physicists and biologists.

Learn this as well...

Advantages of turbulence:

The turbulent agitation in the layers of air near the ground and in the waters of the oceans cause the diffusion of heat and matter, and provide a mechanism for very efficient mixing. Without turbulence, the air at the ground level would either be intolerably hot or very cold and either extremely humid or very dry; and smoke would cling to the ground for days.

Turbulence is also extremely important for bird and insect flights, and for air pollination.

Disadvantages of turbulence:

Turbulence is undesirable in mechanical flights (aircraft), water navigation, ballistics, high-speed cars, etc., because of increased resistance. To reduce turbulence, and hence fluid resistance, these bodies are given a streamlined or aerodynamic shape.

Q. 90. Distinguish between streamline flow and turbulent flow. (2 marks)

Ans. Streamline flow Turbulent flow

1. The steady flow of a fluid, with velocity less than certain critical velocity is called streamline or laminar flow.
2. In a streamline flow, the velocity of a fluid at a given point is always constant.
3. Streamlines do not change and never intersect.
4. The fluid flow is laminar over a surface, and is in the form of coaxial cylinders through a pipe.
1. A non-steady irregular fluid flow in which streamlines and flowtubes change continuously with a velocity greater than certain critical velocity.
2. In a turbulent flow, the velocity of a fluid at any point does not remain constant.
3. Streamlines and flowtubes change continuously.
4. Fluid particles still move in general towards the main direction as before. But now all sorts of secondary motions cause eddies or vortices.

Q. 91. Explain the Reynolds number. OR What is Reynolds number? (3 marks)

Ans. Osborne Reynolds found that if the free-stream velocity of a fluid increases when it moves relative to a solid body, a point is reached where the steady flow becomes turbulent. From experiments, he found that the transition from steady to turbulent flow depends on the value of the quantity \( \frac{\nu_0 d}{\eta / \rho} \), where \( \nu_0 \) is the free-stream velocity, \( d \) is some characteristic dimension of the system, \( \rho \) the density of the fluid and \( \eta \) its coefficient of viscosity.

For a sphere in a fluid stream, \( d \) is its diameter; for water in a pipe, \( d \) is the pipe diameter.

This dimensionless number, defined as \( \text{Re} = \frac{\nu_0 d \rho}{\eta} \), is called the Reynolds number.
In a system of particular geometry, transition from a steady to turbulent flow is given by a certain value of the Reynolds number called the critical Reynolds number. The free-stream velocity for this critical Reynolds number is called the critical velocity, \( v_{\text{critical}} = \frac{nR_e}{\rho d} \). For a given system geometry, the free stream velocity beyond which a streamline flow becomes turbulent is called critical velocity.

Steady flow takes place for Re up to about 1000. For \( 1000 < \text{Re} < 2000 \), there is a transition region in which the flow is extremely sensitive to all sorts of small disturbances. For \( \text{Re} > 2000 \), the flow is completely turbulent.

**Notes:** (1) See Q. 95 for “free-stream velocity”. (2) The dimensionless number is named after Osborne Reynolds (1842 – 1912), British physicist.

Q. 92. Explain the term viscosity. **(2 marks)**

Ans. Suppose a constant tangential force is applied to the surface of a liquid. Under this shearing force, the liquid begins to flow. The motion of a thin layer of the liquid at the surface, relative to a layer below, is opposed by fluid friction. Because of this internal fluid friction, horizontal layers of the liquid flow with varying velocities.

This also happens in a gas. When a solid surface is moved through a gas, a thin layer of the gas moves with the surface. But its motion relative to a layer away is opposed by fluid friction.

The resistance to relative motion between the adjacent layers of a fluid is known as viscosity. It is a property of the fluid. The resistive force in fluid motion is called the viscous drag.

Q. 93. When a liquid contained in a bucket is stirred and left alone, it comes to rest after some time. Why? **(1 mark)**

Ans. This happens due to the internal friction (viscosity) and friction with the walls and bottom of the bucket.

Q. 94. What do you mean by viscous drag? **(1 mark)**

Ans. When a fluid flows past a solid surface, or when a solid body moves through a fluid, there is always a force of fluid friction opposing the motion. This force of fluid friction is called the drag force or viscous drag.

Q. 95. What causes viscous drag in fluids? **(2 marks)**

Ans. In liquids, the viscous drag is due to short range molecular cohesive forces while in gases it is due to collisions between fast moving molecules. For laminar flow in both liquids and gases, the viscous drag is proportional to the relative velocity between the layers, provided the relative velocity is small. For turbulent flow, the viscous drag increases rapidly and is proportional to some higher power of the relative velocity.

Q. 96. Define and explain velocity gradient in a steady flow. **(3 marks)**

Ans. **Definition:** In a steady flow of a fluid past a solid surface, the rate at which the velocity changes with distance within a limiting distance from the surface is called the velocity gradient.

When a fluid flows past a surface with a low velocity, within a limiting distance from the surface, its velocity varies with the distance from the surface, Fig. 2.31. The layer in contact with the surface is at rest relative to the surface. Starting outwards from the surface, the next layer has an extremely small velocity; each successive layer has a slightly higher velocity than its inner neighbour, as shown. Finally, a layer is reached which has approximately the full, or free-stream, velocity \( v_0 \) of the fluid. The situation is reversed if a body is moving in a stationary fluid: the fluid velocity reduces as the distance of a layer from the body increases. Thus, the velocity in each layer increases with its distance from the surface.

Consider the layer of thickness \( dy \) at \( y \) from the solid surface. Let \( v \) and \( v + dv \) be the velocities of
the fluid at the base and upper edge of this layer. The change in velocity across the layer is \( dv \). Therefore, the rate at which the velocity changes between the layers is \( \frac{dv}{dy} \). This is called the velocity gradient.

Q. 97. State and explain Newton’s law of viscosity. 
(3 marks)

Ans. Newton’s law of viscosity: In a steady flow of a fluid past a solid surface, a velocity profile is set up such that the viscous drag per unit area on a layer is directly proportional to the velocity gradient.

When a fluid flows past a solid surface in a streamline flow or when a solid body moves through a fluid, the force of fluid friction opposing the motion is called the viscous drag. The magnitude of the viscous drag of a fluid is given by Newton’s law of viscosity.

If \( \frac{dv}{dy} \) is the velocity gradient, the viscous drag per unit area on a layer,
\[
F = \frac{dv}{dy}
\]

\[
A \propto \frac{dv}{dy}
\]

\[
F = \eta \frac{dv}{dy}
\]

where the constant of proportionality, \( \eta \), is called the coefficient of viscosity of the fluid.

Q. 98. Define coefficient of viscosity. 
(1 mark)

Ans. Coefficient of viscosity: The coefficient of viscosity of a fluid is defined as the viscous drag per unit area acting on a fluid layer per unit velocity gradient established in a steady flow.

Q. 99. Find the dimensions of the coefficient of viscosity. State its SI and CGS units. 
(3 marks)

Ans. By Newton’s law of viscosity,
\[
F = \frac{dv}{dy} = \eta \frac{dv}{dy}
\]

where \( \frac{F}{A} \) is the viscous drag per unit area, \( \frac{dv}{dy} \) is the velocity gradient and \( \eta \) is the coefficient of viscosity of the fluid. Rewriting the above equation as
\[
\eta = \frac{(F/A)}{\left(\frac{dv}{dy}\right)}
\]

\[
\left[\eta\right] = \left[M L^{-1} T^{-1}\right][T^{-1}] = \left[M L^{-3} T^{-1}\right]
\]

SI unit: the pascal-second (abbreviated Pa·s),
1 Pa·s = 1 N·m⁻²·s

CGS unit: dyne·cm⁻²·s, called the poise [symbol P], named after Jean Louis Marie Poiseuille (1799–1869), French physician.

[Note: The most commonly used submultiples are the millipascal-second (mPa·s) and the centipoise (cP). 1 mPa·s = 1 cP.]

Q. 100. Define the SI and CGS units of coefficient of viscosity. 
(2 marks)

Ans. The SI unit of coefficient of viscosity is the pascal-second.

Definition: If a tangential force per unit area of one newton per square metre is required to maintain a difference in velocity of one metre per second between two parallel layers of a fluid in streamline flow separated by one metre, the coefficient of viscosity of the fluid is one pascal-second.

The CGS unit of coefficient of viscosity is the poise.

Definition: If a tangential force per unit area of one dyne per square centimetre is required to maintain a difference in velocity of one centimetre per second between two parallel layers of a fluid in streamline flow separated by one centimetre, the coefficient of viscosity of the fluid is one poise.

Q. 101. Find the conversion factor between the SI and CGS units of coefficient of viscosity using dimensional analysis. 
(3 marks)

Ans. The dimensions of the coefficient of viscosity \( \eta \) are
\[
\left[\eta\right] = \left[M L^{-1} T^{-1}\right]
\]

The SI and CGS units of coefficient of viscosity are the pascal-second and poise, respectively.

1 Pa·s = 1 N·m⁻²·s = 1 kg·m⁻¹·s⁻¹
1 P = 1 dyn·cm⁻²·s = 1 g·cm⁻¹·s⁻¹

Let 1 Pa·s = \( x \)P

\[
\therefore 1 \left[M_1 L_1^{-1} T_1^{-1}\right] = x \left[M_2 L_2^{-1} T_2^{-1}\right]
\]

where subscripts 1 and 2 pertain to SI and CGS units.

\[
\therefore x = \left[\frac{M_1}{M_2}\right] \left[\frac{L_1}{L_2}\right]^{-1} \left[\frac{T_1}{T_2}\right]^{-1}
\]

where subscripts 1 and 2 pertain to SI and CGS units.
Use your brain power

(Textbook page 46)
The CGS unit of viscosity is the poise. Find the relation between the poise and the SI unit of viscosity. Refer to the answer to Q. 99. 1 P = 0.1 Pa·s. 1 mPa.s = 1 cP.

A Microscopic View of Viscosity :

(Textbook page 46)
A classic explanation of viscosity is by using the two-plates model. In this description, a thin layer of fluid sample is placed between two parallel plates, a distance $h$ apart. It is assumed that the fluid has a perfect adhesion onto the two plates—that is, there is a thin boundary layer of fluid clinging to a solid surface in which the fluid is nearly at rest with respect to the surface.

![Fig. 2.32: Two-plates model of viscosity](image)

The lower plate is held stationary and a force $F$ is applied to the upper plate of area $A$, parallel to it in order to avoid exerting a pressure on the fluid. The shear stress is $\sigma = F/A$. Under the applied force, the upper plate accelerates until reaching its constant velocity $v_0$. If under these conditions the fluid enters a state of laminar flow, the speed of each sheet of fluid decreases linearly from the maximum $v_0$ in contact with the upper plate, to zero, in contact with the lower plate. Then, the velocity gradient is $\frac{dv}{dy} = \frac{v_0}{h}$ and the coefficient of viscosity is $\eta = \frac{\sigma}{\frac{dv}{dy}}$.

Viscosity of a gas increases as temperature increases which is opposite with liquids (most liquids become less viscous as temperature increases). This behaviour indicates the mechanism of viscous drag. In a gas, the molecules possess an average molecular momentum and move randomly about at an average speed. As they occasionally collide with each other, molecules possessing greater than average momentum will impart some of that momentum to molecules possessing less than average momentum. As the molecules move with random motion between the layers, colliding molecules exchange their individual momentum. On the average, molecules passing from a layer moving with speed $v$ to the immediate lower layer, imparts more momentum in the direction of $v$. This increases the speed of the molecules of the lower layer. This transfer of momentum continues down through the depth of the fluid to the penultimate layer.

Molecules with lower speed passing from a lower layer to an adjacent upper layer tries to slow it down. The force is required to keep the upper plate moving at a constant velocity due to the internal friction. Thus, the process is dissipative and the energy dissipated increases the thermal energy of the fluid at the cost of mechanical kinetic energy.

In liquids, the molecules are in contact (as opposed to gases where the molecules are far apart), so that there is an additional stronger intermolecular forces between molecules in adjacent layers. Thus, the viscous drag results from a solid-like friction between two layers in relative motion.

Remember this

(Textbook page 47)
Coefficient of viscosity of a fluid changes with change in its temperature. For most liquids, the coefficient of viscosity decreases with increase in their temperature. It probably depends on the fact that at higher temperatures, the molecules are farther apart and the cohesive forces or inter-molecular forces are, therefore, less effective. Whereas, in gases, the coefficient of viscosity increases with the increase in temperature. This is because, at high temperatures, the molecules move faster and collide more often with each other, giving rise to increased internal friction.

Q. 102. State Stokes’ law. (1 mark) Derive Stokes’ law using dimensional analysis. (4 marks)

Ans. Stokes’ law: If a fluid flows past a sphere or a sphere moves through a fluid, for small enough
relative speed $v_o$ for which the flow is streamline, the viscous force on the sphere is directly proportional to the coefficient of viscosity of the fluid $\eta$, the radius $r$ of the sphere and the free-stream velocity $\vec{v}_0$.

Consider a fluid in steady flow moving relative to a sphere of radius $r$ with a velocity $\vec{v}_0$, called the free-stream velocity. The fluid in immediate contact with the sphere is essentially at rest. The speed of the fluid increases with increasing distance from the sphere. It is found that this entire velocity profile is confined to a very shallow layer, called the boundary layer, adjacent to the sphere and that outside this boundary layer the fluid flows at its free-stream velocity. The fluid in immediate contact with the sphere is essentially at rest. The speed of the sphere falls through the fluid under gravity, its speed increases. According to Stokes’ law, the magnitude of the viscous force depends on $\eta$, $v$, and $v_0$, we can write

$$f = -b \cdot \eta \cdot r^\beta \cdot v_0^\gamma$$

where $b$ is a dimensionless proportionality constant.

Therefore,

$$[f] = [\eta] \cdot [r] \cdot [v_0]$$

With $[\eta] = ML^{-1}T^{-1}$, $[r] = L$, $[f] = ML^1T^{-2}$ and $[v_0] = LT^{-1}$, we get,

$$ML^1T^{-2} = (ML^{-1}T^{-1})\cdot (LT^{-1})^\beta \cdot (LT^{-1})^\gamma$$

$$\therefore ML^1T^{-2} = M^\beta \cdot L^{-\gamma} \cdot T^{-\beta+\gamma}$$

Homogeneity of the above dimensional equation requires that $x = 1$, $-x + \beta + \gamma = 1$ and $-x + \gamma = -2$

On solving, we get,

$$\gamma = -x + 2 = -1 + 2 = 1$$

$$\beta = 1 + x - \gamma = 1 + 1 = 1$$

Thus, $x = \beta = \gamma = 1$

$$\therefore f = -b \eta rv_0$$

Inserting the value of $b$ from theory and experiments,

$$f^2 = -6\pi \eta rv_0^2$$

This is called Stokes’ law, after Sir George Gabriel Stokes (1819–1903), British physicist and mathematician.

Q. 103. State Stokes’ law and an expression for the magnitude of the viscous force. (2 marks)

Ans. Refer to the answer to Q. 102 for the definition.

The magnitude of the viscous force on a sphere of radius $r$ is

$$f = 6\pi \eta rv_0$$

where $v_0$ = the free-stream velocity of the fluid flowing past the sphere and $\eta$ = the coefficient of viscosity of the fluid. The constant $6\pi$ is obtained from theory and experiment.

Q. 104. Explain what is meant by the terminal speed of a body falling through a viscous fluid. Hence obtain the expression for the terminal speed. (2 marks)

Ans. Consider a small sphere of radius $r$, falling through a fluid with coefficient of viscosity $\eta$. Initially, as the sphere falls through the fluid under gravity, its speed increases. According to Stokes’ law, the magnitude $f$ of the viscous force on the falling sphere is proportional to its speed $v$. The direction of this force is upward since the velocity of the sphere is downward. Also, the fluid exerts an upthrust or buoyant force on the sphere. As soon as the speed reaches a value, $v_t$, the magnitude of $f$ becomes equal to that of the gravitational force on the sphere minus the upthrust. Then, the net force acting on the sphere becomes zero. Its subsequent downward motion is at this constant speed. This is called its terminal speed: $v_t$ represents the highest speed which a body can attain when freely falling through a fluid with coefficient of viscosity $\eta$.

If the sphere and the fluid have densities $\rho$ and $\rho_L$, respectively, the total downward force on the sphere is the sum of the downward gravitational force and the upward buoyant force.
Solution:

\[ f = 6\pi \eta rv_i \]
\[ \therefore 6\pi \eta rv_i = \frac{4}{3} \pi r^3 (\rho - \rho_L) g \]
\[ v_i = \frac{2 r^2 (\rho - \rho_L) g}{\eta} \]

[Note: Theoretically, \( v \rightarrow v_i \) as time \( t \rightarrow \infty \). In practice, if \( \eta \) is appreciable, then \( v \) tends to \( v_i \) in a very small time interval.]

Remember this

(Textbook page 48)
The velocity with which an object can move through a viscous fluid is always less than or equal to the terminal velocity in that fluid for that object.

Solved Problems 2.5–2.7.1

Q. 105. Solve the following:

(1) The relative velocity between two layers of a fluid, separately by 0.1 mm, is 2 cm/s. Calculate the velocity gradient. (1 mark)

Solution:

Data: \( dy = 0.1 \text{ mm} = 10^{-2} \text{ cm}, \frac{dv}{dy} = 2 \text{ cm/s} \)

The velocity gradient, \( \frac{dv}{dy} = \frac{2 \text{ cm/s}}{10^{-2} \text{ cm}} = 200 \text{ s}^{-1} \)

(2) Calculate the force required to move a flat glass plate of area 10 cm\(^2\) with a uniform velocity of 1 cm/s over the surface of a liquid 2 mm thick, if the coefficient of viscosity of the liquid is 2 Pa·s. (2 marks)

Solution:

Data: \( \eta = 2 \text{ Pa·s}; A = 10 \text{ cm}^2 = 10^{-3} \text{ m}^2; \)
\( dv = 1 \text{ cm/s} = 0.01 \text{ m/s}; dy = 2 \text{ mm} = 2 \times 10^{-3} \text{ m} \)

According to Newton’s formula,

viscous force \( f = \eta A \frac{dv}{dy} \)

\[ = \frac{(2 \text{ Pa·s}) (10^{-3} \text{ m}^2) (0.01 \text{ m/s})}{2 \times 10^{-3} \text{ m}} \]
\[ = 0.01 \text{ N} \]

This force retards the motion of the glass plate. Therefore, in order to keep the plate moving with a uniform velocity, an equal force must be exerted on the plate in the forward direction.

The required force to move the glass plate is 0.01 N.

(3) A metal plate of length 10 cm and breadth 5 cm is in contact with a layer of oil 1 mm thick. The horizontal force required to move it with a velocity of 4 cm/s along the surface of the oil is 0.32 N. Find the coefficient of viscosity of the oil. Also express it in poise. (3 marks)

Solution:

Data: \( A = 10 \text{ cm} \times 5 \text{ cm} = 50 \text{ cm}^2 = 5 \times 10^{-3} \text{ m}^2, \)
\( y = 1 \text{ mm} = 10^{-3} \text{ m}, v_0 = 4 \text{ cm/s} = 4 \times 10^{-2} \text{ m/s}, \)
\( F = 0.32 \text{ N} \)

Velocity gradient, \( \frac{v_0}{y} = \frac{4 \times 10^{-2} \text{ m/s}}{10^{-3} \text{ m}} = 40 \text{ s}^{-1} \)

Viscous force, \( F = \eta A \frac{v_0}{y} \)

\[ \therefore \text{The coefficient of viscosity is} \]
\[ \eta = \frac{F}{A (\frac{v_0}{y})} \]
\[ = \frac{0.32 \text{ N}}{(5 \times 10^{-3} \text{ m}^2)(40 \text{ s}^{-1})} \]
\[ = \frac{320}{200} = 1.6 \text{ N·s/m}^2 \]

Since 1 N·s/m\(^2\) = 10 poise,
\[ \eta = 16 \text{ poise} \]

(4) A horizontal force of 1 N is required to move a metal plate of area \( 10^{-2} \text{ m}^2 \) with a velocity of \( 2 \times 10^{-2} \text{ m/s}, \) when it rests on a layer of oil 1.5 \( \times \) \( 10^{-3} \text{ m} \) thick. Find the coefficient of viscosity of oil. (3 marks)

Solution:

Data: \( F = 1 \text{ N}, A = 10^{-2} \text{ m}^2, v_0 = 2 \times 10^{-2} \text{ m/s}, \)
\( y = 1.5 \times 10^{-3} \text{ m} \)

Velocity gradient, \( \frac{dv}{dy} = \frac{2 \times 10^{-2} \text{ m/s}}{1.5 \times 10^{-3} \text{ m}} = \frac{40}{3} \text{ s}^{-1} \)
Viscous force, \( F = \eta \frac{dv}{dy} \)

\[ \therefore \text{ The coefficient of viscosity is} \]
\[ \eta = \frac{F}{A(\frac{dv}{dy})} = \frac{1 \text{ N}}{\left(10^{-2} \text{ m}^2\right) \left(40/3 \text{ s}^{-1}\right)} = \frac{30}{4} = 7.5 \text{ Pa·s} \]

\( \star \) (5) Calculate the viscous force acting on a rain drop of diameter 1 mm, falling with a uniform velocity of 2 m/s through air. The coefficient of viscosity of air is \( 1.8 \times 10^{-5} \text{ N·s/m}^2 \).

**Solution:**

**Data:**
- \( d = 1 \text{ mm} \), \( v_0 = 2 \text{ m/s} \)
- \( \eta = 1.8 \times 10^{-5} \text{ N·s/m}^2 \)
- \( r = \frac{d}{2} = 0.5 \text{ mm} = 5 \times 10^{-4} \text{ m} \)

By Stokes’ law, the viscous force on the raindrop is

\[ f = 6\pi \eta v_0 r \]
\[ = 6 \times 3.142 \left(1.8 \times 10^{-5} \text{ N·s/m}^2 \times 5 \times 10^{-4} \text{ m}\right) \]
\[ = 3.394 \times 10^{-7} \text{ N} \]

\( \star \) (6) A spherical liquid drop of diameter \( 2 \times 10^{-4} \text{ m} \) is falling with a constant velocity through air, under gravity. If the density of the liquid is 500 kg/m\(^3\) and the coefficient of viscosity of air is \( 2 \times 10^{-5} \text{ Pa·s} \), determine the terminal velocity of the drop and the viscous force acting on it. Ignore the density of air.

**Solution:**

**Data:**
- \( r = 1 \times 10^{-4} \text{ m} \), \( \rho = 500 \text{ kg/m}^3 \), \( \rho_{\text{air}} \ll \rho \)
- \( \eta = 2 \times 10^{-5} \text{ Pa·s} \)

(i) The terminal velocity,

\[ v_t = \frac{2r^2g(\rho - \rho_{\text{air}})}{9\eta} \approx \frac{2r^2g\rho}{9\eta} \quad \left( \therefore \rho_{\text{air}} \ll \rho \right) \]
\[ = \frac{2 \left(1 \times 10^{-4} \text{m}\right)^2 (9.8 \text{ m/s}^2)(500 \text{ kg/m}^3)}{9 \left(2 \times 10^{-5} \text{ Pa·s}\right)} \]
\[ = 0.544 \text{ m/s} \]

(ii) The viscous force,

\[ F = 6\pi \eta v_t \]
\[ = 6 \times 3.142 \times \left(2 \times 10^{-5} \text{ Pa·s}\right) (0.544 \text{ m/s}) \]
\[ = 2.051 \times 10^{-8} \text{ N} \]

\( \star \) (7) With what terminal velocity will an air bubble 0.4 mm in diameter rise in a liquid of viscosity 0.1 Ns/m\(^2\) and specific gravity (relative density) 0.9? Density of air is 1.29 kg/m\(^3\).

**Solution:**

**Data:**
- \( d = 0.4 \text{ mm} \), \( \eta = 0.1 \text{ Pa·s} \), \( \rho_L = 0.9 \times 10^3 \text{ kg/m}^3 \)
- \( = 900 \text{ kg/m}^3 \), \( \rho_{\text{air}} = 1.29 \text{ kg/m}^3 \), \( g = 9.8 \text{ m/s}^2 \).

Since the density of air is less than that of oil, the air bubble will rise up through the liquid. Hence, the viscous force is downward. At terminal velocity, this downward viscous force is equal in magnitude to the net upward force.

Viscous force = buoyant force – gravitational force

\[ \therefore 6\pi \eta v_t = \frac{4}{3} \pi r^3 (\rho_L - \rho_{\text{air}})g \]

\[ \therefore \text{ The terminal velocity,} \]
\[ v_t = \frac{2r^2g(\rho_L - \rho)}{9\eta} \]
\[ = \frac{2 \left(2 \times 10^{-4} \text{ m}\right)^2 (9.8 \text{ m/s}^2) (900 - 1.29)}{9 \left(0.1 \text{ Pa·s}\right)} \]
\[ = \frac{2 \times 4 \times 9.8 \times 898.7 \times 10^{-8}}{0.9} \]
\[ = 7.829 \times 10^4 \times 10^{-8} \]
\[ = 7.829 \times 10^{-4} \text{ m/s} = 0.7829 \text{ mm/s (upward)} \]

\( \star \) (8) If the speed at which water flows through a long cylindrical pipe of radius 8 mm is 10 cm/s, find the Reynolds number. [Density of water = 1 g/cm\(^3\), coefficient of viscosity of water = 0.01 poise]

**Solution:**

**Data:**
- \( v_0 = 10 \text{ cm/s} \), \( \rho = 1 \text{ g/cm}^3 \), \( r = 8 \text{ mm} \)
- \( \therefore d = 2r = 16 \text{ mm} = 1.6 \text{ cm} \), \( \eta = 0.01 \text{ poise} \)

The Reynolds number, \( \text{Re} = \frac{v_0 d}{\eta} \)
\[ = \frac{(10 \text{ cm/s})(1 \text{ g/cm}^3)(1.6 \text{ cm})}{0.01 \text{ P}} = 1600 \]

2.8 Equation of continuity

Q. 106. Define volume flow rate or volume flux. Explain how it is related to the velocity of fluid.

**OR**

What is the difference between flow rate and fluid velocity? How are they related?
Ans. Definition: The volume of fluid passing by a given location per unit time through an area is called the **volume flow rate**, or simply **flow rate**, \( Q = \frac{dV}{dt} \)

![Fig. 2.34: A volume \( Ax \) of an ideal fluid flowing past P in a pipe in time \( t \)](image)

Consider an ideal fluid flowing with velocity \( v \) through a uniform flow tube of cross section \( A \). If, as shown in Fig. 2.34, the shaded cylinder of fluid of length \( x \) and volume \( V \) flows past point P in time \( t \),

\[
v = \frac{dx}{dt} \quad \text{and} \quad V = Ax
\]

Then, the volume flow rate is

\[
Q = \frac{dV}{dt} = \frac{d}{dt} (AX) = A \frac{dx}{dt} = Av
\]

which is the required relation.

Q. 107. State the SI unit for volume flow rate. (**2 marks**)

**Ans.**

The SI unit for volume flow rate is the cubic metre per second (m\(^3\)/s).

**Note:** Another common unit accepted in SI is the litre per minute (L/min). 1 L = 10\(^{-3}\) m\(^3\) = 10\(^4\) cm\(^3\). An old non-SI unit from FPS system still used is the cubic feet per second (symbol, cusec).

**Do you know?**

(Textbook page 49)

The rate at which water is released from a dam is specified in Thousand Million Cubic feet per second (TMC). 1 TMC = 10\(^9\) cusecs. 1 cusec = 2.832 \times 10\(^{-2}\) m\(^3\)/s = 28.32 L/s = 1700 L/min.

Q. 108. Define mass flow rate or mass flux. Explain how it is related to the velocity of fluid. (**2 marks**)

**Ans.**

**Definition:** The mass of fluid passing by a given point per unit time through an area is called the **mass flow rate**, \( dm/dt \).

Refer to Fig. 2.34.

Consider an ideal fluid of density \( \rho \) flowing with velocity \( v \) through a uniform flow tube of cross section \( A \). If, as shown in Fig. 2.34, the shaded cylinder of fluid of length \( x \) and volume \( V \) flows past point P in time \( t \),

\[
v = \frac{dx}{dt} \quad \text{and} \quad V = Ax
\]

Then, the mass flow rate is

\[
\frac{dm}{dt} = \frac{d}{dt} (\rho V) = \rho \frac{dV}{dt} = \rho Av
\]

which is the required relation.

Q. 109. Explain the continuity condition for a flow tube. Show that the flow speed is inversely proportional to the cross-sectional area of a flow tube. (**3 marks**)

**Ans.**

Consider a fluid in steady or streamline flow. The velocity of the fluid within a flow tube, while everywhere parallel to the tube, may change its magnitude. Suppose the velocity is \( \vec{v}_1 \) at point P and \( \vec{v}_2 \) at point Q. If \( A_1 \) and \( A_2 \) are the cross-sectional areas of the tube and \( \rho_1 \) and \( \rho_2 \) are the densities of the fluid at these two points, the mass of the fluid passing per unit time across \( A_1 \) is \( A_1 \rho_1 v_1 \) and that passing across \( A_2 \) is \( A_2 \rho_2 v_2 \). Since no fluid can enter or leave through the boundary of the tube, the conservation of mass requires

\[
A_1 \rho_1 v_1 = A_2 \rho_2 v_2 \quad \text{... (1)}
\]

Equation (1) is called the equation of continuity of flow. It holds true for a compressible fluid, (like all gases) for which the density of the fluid may differ from point to point in a tube of flow. For an incompressible fluid (like all liquids), \( \rho_1 = \rho_2 \) and Eq. (1) takes the simpler form

\[
A_1 v_1 = A_2 v_2 \quad \text{... (2)}
\]

\[
\therefore \quad \frac{v_1}{v_2} = \frac{A_2}{A_1} \quad \text{... (3)}
\]

that is, the flow speed is inversely proportional to the cross-sectional area of a flow tube. Where the
area is large, the speed of flow is small, and vice versa.

Equations (2) is the equation of continuity for an incompressible fluid for which density is constant throughout.

Q. 110. Explain why flow speed is greatest where streamlines are closest together. (1 mark)
Ans. By the equation of continuity, the flow speed is inversely proportional to the area of cross section of a flow tube. Where the area of cross section is small, i.e., streamlines are close, the flow speed is large and vice versa.

★ Q. 111. Obtain an expression for conservation of mass starting from the equation of continuity. (3 marks)
Ans. Refer to Fig. 2.35
Consider a fluid in steady or streamline flow, that is its density is constant. The velocity of the fluid within a flow tube, while everywhere parallel to the tube, may change its magnitude. Suppose the velocity is \( \vec{v}_1 \), at point P and \( \vec{v}_2 \) at point Q. If \( A_1 \) and \( A_2 \) are the cross-sectional areas of the tube at these two points, the volume flux across \( A_1, \frac{d}{dt}(V_1) = A_1v_1 \) and that across \( A_2, \frac{d}{dt}(V_2) = A_2v_2 \)

By the equation of continuity of flow for a fluid,
\[ A_1v_1 = A_2v_2 \]
i.e., \[ \frac{d}{dt}(V_1) = \frac{d}{dt}(V_2) \]
If \( \rho_1 \) and \( \rho_2 \) are the densities of the fluid at P and Q, respectively, the mass flux across \( A_1, \frac{d}{dt}(m_1) = \frac{d}{dt}(\rho_1V_1) = A_1\rho_1v_1 \) and that across \( A_2, \frac{d}{dt}(m_2) = \frac{d}{dt}(\rho_2V_2) = A_2\rho_2v_2 \)
Since no fluid can enter or leave through the boundary of the tube, the conservation of mass requires the mass fluxes to be equal, i.e.,
\[ \frac{d}{dt}(m_1) = \frac{d}{dt}(m_2) \]
i.e., \( A_1\rho_1v_1 = A_2\rho_2v_2 \)
i.e., \( Apv = \) constant
which is the required expression.

★ Q. 112. Why does velocity increase when water flowing in broader pipe enters a narrow pipe? (2 marks)
Ans. When a tube narrows, the same volume occupies a greater length, as schematically shown in Fig. 2.36. \( A_1 \) is the cross section of the broader pipe and that of narrower pipe is \( A_2 \). By the equation of continuity, 
\[ v_2 = (A_1/A_2)v_1 \]

![Fig. 2.36](Textbook page 49)
A water pipe with a diameter of 5.0 cm is connected to another pipe of diameter 2.5 cm. How would the speeds of the water flow compare?
Ans. Water is an incompressible fluid (almost). Then, by the equation of continuity, the ratio of the speeds, is 
\[ \frac{v_1}{v_2} = \frac{A_2}{A_1} = \left(\frac{d_1}{d_2}\right)^2 = \left(\frac{5}{2.5}\right)^2 = \frac{1}{4} \]

★ Q. 113. You can squirt water a considerably greater distance by placing your thumb over the end of a garden hose. Explain. (1 mark)
Ans. Placing one’s thumb over the end of a garden hose constricts the open end. By the continuity condition, the speed of water increases as it passes through the constriction. Hence, water squirts out and reaches a longer distance.

★ Solved Problems 2.8

Q. 114. Solve the following:

1) A liquid is flowing through a horizontal pipe of varying cross section. At a certain point, where the diameter of the pipe is 5 cm, the flow velocity is 0.25 m/s. What is the flow velocity where the diameter is 1 cm? (2 marks)
Solution:

Data: \( d_1 = 5 \text{ cm}, v_1 = 0.25 \text{ m/s}, d_2 = 1 \text{ cm} \)

According to the equation of continuity of flow,

\[ A_1 \rho_1 v_1 = A_2 \rho_2 v_2 \]

where \( A_1 \) and \( \rho_1 \) are the cross-sectional area and density of the liquid where the flow velocity is \( v_1 \); \( A_2 \) and \( \rho_2 \) are the corresponding quantities where the flow velocity is \( v_2 \).

Assuming the liquid is incompressible,

\[ \rho_1 = \rho_2 \]

\[ \therefore A_1 v_1 = A_2 v_2 \]

\[ \therefore v_2 = \frac{A_1}{A_2} v_1 \]

\[ = \left( \frac{\pi d_1^2/4}{\pi d_2^2/4} \right) v_1 = \left( \frac{d_1}{d_2} \right)^2 v_1 \]

\[ = \left( \frac{5 \text{ cm}}{1 \text{ cm}} \right)^2 \times (0.25 \text{ m/s}) \]

\[ = 25 \times 0.25 = 6.25 \text{ m/s} \]

★ (2) The speed of water through a pipe of internal diameter 10 cm is 2 m/s. What should be the internal diameter of nozzle of the pipe if the speed of water at nozzle is 4 m/s? (2 marks)

Solution:

Data: \( d_1 = 10 \text{ cm} = 0.1 \text{ m}, v_1 = 2 \text{ m/s}, v_2 = 4 \text{ m/s} \)

By the equation of continuity, the ratio of the speed is

\[ \frac{v_1}{v_2} = \frac{A_2}{A_1} = \left( \frac{d_2}{d_1} \right)^2 \]

\[ \therefore \frac{d_2}{d_1} = \sqrt{\frac{v_1}{v_2}} = \sqrt{\frac{2}{4}} = 1 \sqrt{\frac{1}{2}} = 0.707 \]

\[ \therefore d_2 = 0.707 d_1 = 0.707(0.1 \text{ m}) = 0.0707 \text{ m} \]

Unit

2.9 Bernoulli equation, Applications

Do you know? (Textbook page 50)

1. How does an aeroplane take off?
   Refer to the answer to Q. 121.

2. Why do racer cars and birds have typical shape?
   The streamline shape of cars and birds reduce drag.

3. Have you experienced a sideways jerk while driving a two wheeler when a heavy vehicle overtakes you?

   Suppose a truck passes a two-wheeler or car on a highway. Air passing between the vehicles flows in a narrower channel and must increase its speed according to Bernoulli’s principle causing the pressure between them to drop. Due to greater pressure on the outside, the two-wheeler or car veers towards the truck.

   ![Diagram](image)

When two ships sail parallel side-by-side within a distance considerably less than their lengths, since ships are widest toward their middle, water moves faster in the narrow gap between them. As water velocity increases, the pressure in between the ships decreases due to the Bernoulli effect and draws the ships together. Several ships have collided and suffered damage in the early twentieth century. Ships performing At-sea refuelling or cargo transfers performed by ships is very risky for the same reason.

4. Why does dust get deposited only on one side of the blades of a fan?

   Blades of a ceiling/table fan have uniform thickness (unlike that of an aerofoil) but are angled (cambered) at 8° to 12° (optimally, 10°) from their plane. When they are set rotating, this camber causes the streamlines above/behind a fan blade to detach away from the surface of the blade creating a very low pressure on that side. The lower/front streamlines however follow the blade surface. Dust particles
stick to a blade when it is at rest as well as when in motion both by intermolecular force of adhesion and due to static charges. However, they are not dislodged from the top/behind surface because of complete detachment of the streamlines. The lower/front surface retains some of the dust because during motion, a thin layer of air remains stationary relative to the blade.

5. Why helmets have specific shape?
Air drag plays a large role in slowing bike riders (especially, bicycle) down. Hence, a helmet is aerodynamically shaped so that it does not cause too much drag.

Q. 115. State Bernoulli’s principle.  
(1 mark)
Ans. Where the velocity of an ideal fluid in streamline flow is high, the pressure is low, and where the velocity of a fluid is low, the pressure is high. OR
At every point in the streamline flow of an ideal (i.e., nonviscous and incompressible) fluid, the sum of the pressure energy, kinetic energy and potential energy of a given mass of the fluid is constant at every point.

[Note: The above principle is equivalent to a statement of the law of conservation of mechanical energy as applied to fluid mechanics. It was published in 1738 by Daniel Bernoulli (1700–82), Swiss mathematician.]

Q. 116. Explain Bernoulli’s equation of fluid flow. 
(4 marks)
Ans. Consider an ideal fluid incompressible and nonviscous of density \( \rho \) flowing along a flow tube of varying cross section. The system under consideration is the flow tube between points 1 and 2, and the Earth (Fig. 2.38). From the continuity equation it follows that pressure and speed must be different in regions of different cross section. If the height also changes, there is an additional pressure difference.

The fluid enters the system at point 1 through a surface of cross section \( A_1 \) at speed \( v_1 \). The point 1 lies at a height \( h_1 \), with respect to an arbitrary reference level \( y = 0 \), and the local pressure there is \( p_1 \). The fluid leaves the system at point 2 where the corresponding quantities are \( A_2 \), \( v_2 \), \( h_2 \) and \( p_2 \).

Consider a small fluid element, of volume \( \Delta V \) and mass \( \Delta m = \rho \Delta V \), that enters at point 1 and leaves at point 2 during small time interval \( \Delta t \). In the absence of internal fluid friction, it can be shown that the work done on the fluid element by the surrounding fluid is

\[ \Delta W = (p_1 - p_2) \Delta V \]

This is sometimes called the pressure energy. During \( \Delta t \), the changes in the kinetic energy and potential energy are

\[ \Delta KE = \frac{1}{2} \Delta m (v_2^2 - v_1^2) = \frac{1}{2} \rho \Delta V (v_2^2 - v_1^2) \]

\[ \Delta PE = \Delta mg(h_2 - h_1) = \rho \Delta V g (h_2 - h_1) \]

Since \( \Delta W \) is the work done by a non-conservative force,

\[ \Delta W = \Delta KE + \Delta PE \]

\[ \therefore \ p_1 - p_2 = \frac{1}{2} \rho (v_2^2 - v_1^2) + \rho g (h_2 - h_1) \]

\[ \therefore \ p_1 + \frac{1}{2} \rho v_1^2 + \rho gh_1 = p_2 + \frac{1}{2} \rho v_2^2 + \rho gh_2 \]

or \( p + \frac{1}{2} \rho v^2 + \rho gh = \text{constant} \) ... (2)

This is known as Bernoulli’s equation.

[Notes: Equation (1) can be rewritten as

\[ p_1 \Delta V + \frac{1}{2} \rho \Delta V v_1^2 = p_2 \Delta V + \frac{1}{2} \rho \Delta V v_2^2 + \rho \Delta Vgh_1 \]

or \( \rho \Delta V + \frac{1}{2} \rho \Delta V v^2 + \rho \Delta Vgy = \text{constant} \) ... (3)

i.e., pressure energy + KE + PE = constant

Dividing Eq. (3) by \( \Delta m = \rho \Delta V \),

\[ \frac{p}{\rho} + \frac{1}{2} v^2 + gy = \text{constant} \]

i.e., pressure energy per unit mass + KE per unit mass + PE per unit mass = constant, which is Bernoulli’s principle. Note that in writing
\[ \Delta W = \Delta KE + \Delta PE \], we have assumed the principle of conservation of energy.

Dimensionally, pressure is energy per unit volume. Both terms on the right side of Eq. (2) also have the same dimensions. Hence, the term \( (p_1 - p_2) \) is often referred to as pressure energy per unit volume or pressure head. The first term on the right, \( \frac{1}{2} \rho (v_2^2 - v_1^2) \), is called the velocity head and the second term, \( \rho g (h_2 - h_1) \), is called the potential head.

Q. 117. What is the basis of Bernoulli’s principle?

Ans. Conservation of energy.

Use your brain power

(Textbook page 52)

Does the Bernoulli’s equation change when the fluid is at rest? How?

Bernoulli’s principle is for fluids in motion. Hence, it is pointless to apply it to a fluid at rest. Nevertheless, for a fluid at rest, the Bernoulli equation gives the pressure difference due to a liquid column.

For a static fluid, \( v_1 = v_2 = 0 \). Bernoulli’s equation in that case is \( p_1 + \rho g h_1 = p_2 + \rho g h_2 \).

Further, taking \( h_2 \) as the reference height of zero, i.e., by setting \( h_2 = 0 \), we get \( p_2 = p_1 + \rho g h_1 \).

This equation tells us that in static fluids, pressure increases with depth. As we go from point 1 to point 2 in the fluid, the depth increases by \( h \) and consequently, \( p_2 \) is greater than \( p_1 \) by an amount \( \rho g h_1 \). In the case, \( p_1 = p_0 \), the atmospheric pressure at the top of the fluid, we get the familiar gauge pressure at a depth \( h_1 = \rho g h_1 \). Thus, Bernoulli’s equation confirms the fact that the pressure change due to the weight of a fluid column of length \( h \) is \( \rho g h \).

Q. 118. State the limitations of Bernoulli’s principle.

(2 marks)

Ans. Limitations: Bernoulli’s principle and his equation for fluid flow is valid only for

1. an ideal fluid, i.e., one that is incompressible and nonviscous, so that the density remains constant throughout a flow tube and there is no viscous drag which results in energy dissipation or loss,

2. streamline flow.

Q. 119. State the applications of Bernoulli’s principle.

(2 marks)

Ans. Applications:

1. Venturi meter: It is a horizontal constricted tube that is used to measure flow speed in a gas.

2. Atomizer: It is a hydraulic device used for spraying insecticide, paint, air perfume, etc.

3. Aerofoil: The aerofoil shape of the wings of an aircraft produces aerodynamic lift.

4. Bunsen’s burner: Bernoulli effect is used to admit air into the burner to produce an oxidising flame.

Q. 120. State the law of efflux. Derive an expression for the speed of efflux for a tank discharging through an opening at a depth \( h \) below the liquid surface. Hence or otherwise show that the speed of efflux for an open tank is \( \sqrt{2gh} \).

Ans. Law of efflux (Torricelli’s theorem): The speed of efflux for an open tank through an orifice at a depth \( h \) below the liquid surface is equal to the speed acquired by a body falling freely through a vertical distance \( h \).

Consider a tank with cross-sectional area \( A_1 \) holding a static liquid of density \( \rho \). The tank discharges through an opening (of cross-sectional area \( A_2 \)) in the side wall at a depth \( h \) below the liquid surface. The flow speed at which the liquid leaves the tank is called the speed of efflux.

The pressure at point 2 it is the atmospheric pressure \( p_0 \). Let the pressure of the air above the liquid at point 1 be \( p \). We assume that the tank is large in cross section compared to the opening \( (A_1 \gg A_2) \), so that the upper surface of the liquid will drop very slowly. That is, we may regard the
liquid surface to be approximately at rest \((v_1 \approx 0)\). Bernoulli’s equation, in usual notation, states
\[
p_1 + \frac{1}{2} \rho v_1^2 + \rho gy_1 = p_2 + \frac{1}{2} \rho v_2^2 + \rho gy_2
\]
Substituting \(p_1 = p, \ p_2 = p_\infty, \ v_1 = 0 \) and \((y_1 - y_2) = h\),
\[
v_2^2 = 2 \frac{p - p_\infty}{\rho} + 2gh
\]
If the tank is open to the atmosphere, then \(p = p_\infty\),
\[
v_2 = \sqrt{2gh}
\]
which is the law of efflux.

[Note : For an open tank, the speed of the liquid, \(v_\infty\), leaving a hole a distance \(h\) below the surface is equal to that acquired by an object falling freely through a vertical distance \(h\).]

Q. 121. What is a Venturi tube? Explain the working of a Venturi tube.

OR

What is a Venturi meter? Briefly explain its use to determine the flow rate in a pipe. (4 marks)

Ans. A Venturi meter is a horizontal constricted tube that is used to measure the flow speed through a pipeline. The constricted part of the tube is called the throat. Although a Venturi meter can be used for a gas, they are most commonly used for liquids. As the fluid passes through the throat, the higher speed results in lower pressure at point 2 than at point 1. This pressure difference is measured from the difference in height \(h\) of the liquid levels in the U-tube manometer containing a liquid of density \(\rho_m\) (Fig. 2.40). The following treatment is limited to an incompressible fluid.

Let \(A_1\) and \(A_2\) be the cross-sectional areas at points 1 and 2, respectively. Let \(v_1\) and \(v_2\) be the corresponding flow speeds. \(\rho\) is the density of the fluid in the pipeline. By the equation of continuity,
\[
v_1 A_1 = v_2 A_2 \tag{1}
\]
Since the meter is assumed to be horizontal, from Bernoulli’s equation we get,
\[
p_1 + \frac{1}{2} \rho v_1^2 = p_2 + \frac{1}{2} \rho v_2^2
\]
\[\therefore \ p_1 + \frac{1}{2} \rho v_1^2 = p_2 + \frac{1}{2} \rho v_2^2 \left(\frac{A_1}{A_2}\right)^2 \tag{from Eq. (1)}\]
\[\therefore \ p_1 - p_2 = \frac{1}{2} \rho v_1^2 \left(\frac{(A_1)^2}{A_2^2} - 1\right) \tag{2}\]

The pressure difference is equal to \(\rho_m g h\), where \(h\) is the differences in liquid levels in the manometer. Then,
\[
\rho_m g h = \frac{1}{2} \rho \left(\frac{(A_1)^2}{A_2^2} - 1\right)
\]
\[\therefore \ v_1 = \sqrt{\frac{2 \rho_m g h}{\rho \left(\frac{(A_1)^2}{A_2^2} - 1\right)}} \tag{3}\]

Equation (3) gives the flow speed of an incompressible fluid in the pipeline. The flow rates of practical interest are the mass and volume flow rates through the meter.

Volume flow rate = \(A_1 v_1\)
and mass flow rate = \(\text{density} \times \text{volume flow rate} = \rho A_1 v_1\)

[Note : When a Venturi meter is used in a liquid pipeline, the pressure difference is measured from the difference in height \(h\) of the levels of the same liquid in the two vertical tubes, as shown in Fig. 2.40. Then, the pressure difference is equal to \(\rho g h\).]
\[ \rho g h = \frac{1}{2} \rho v_1^2 \left[ \left( \frac{A_1}{A_2} \right)^2 - 1 \right] \]

\[ \therefore v_1 = \sqrt{\frac{2gh}{\left[ \left( \frac{A_1}{A_2} \right)^2 - 1 \right]}} \quad \text{... (4)} \]

The flow meter is named after Giovanni Battista Venturi (1746 – 1822), Italian physicist.

**Q. 122. Why does the speed of a liquid increase and its pressure decrease when a liquid passes through a constriction in a horizontal pipe?**  
(3 marks)

**Ans.** Refer Fig. 2.41.

Consider a horizontal constricted tube.

Let \( A_1 \) and \( A_2 \) be the cross-sectional areas at points 1 and 2, respectively. Let \( v_1 \) and \( v_2 \) be the corresponding flow speeds. \( \rho \) is the density of the fluid in the pipeline. By the equation of continuity,

\[ v_1A_1 = v_2A_2 \quad \text{... (1)} \]

\[ \therefore \frac{v_2}{v_1} = \frac{A_1}{A_2} > 1 \quad (\therefore A_1 > A_2) \]

Therefore, the speed of the liquid increases as it passes through the constriction. Since the meter is assumed to be horizontal, from Bernoulli’s equation we get,

\[ p_1 = \frac{1}{2} \rho v_1^2 = p_2 + \frac{1}{2} \rho v_2^2 \]

\[ \therefore p_1 + \frac{1}{2} \rho v_1^2 = p_2 + \frac{1}{2} \rho v_2^2 \left( \frac{A_1}{A_2} \right)^2 \quad \text{[from Eq. (1)]} \]

\[ \therefore p_1 - p_2 = \frac{1}{2} \rho v_1^2 \left[ \left( \frac{A_1}{A_2} \right)^2 - 1 \right] \quad \text{... (2)} \]

Again, since \( A_1 > A_2 \), the bracketed term is positive so that \( p_1 > p_2 \). Thus, as the fluid passes through the constriction or throat, the higher speed results in lower pressure at the throat.

**Q. 123. Explain aerodynamic lift on the wings of an aeroplane.**  

**OR**

**Explain why the upper surface of the wings of an aeroplane is made convex and the lower surface concave.**  
(2 marks)

**Ans.** An aeroplane wing has a special characteristic aerodynamic shape called an aerofoil. An aerofoil is convex on the top and slightly concave on the bottom. Its leading edge is well rounded while the trailing edge is sharp. As an aeroplane moves through air, the aerofoil shape makes the air moving over the top and along the bottom of a wing in a certain way.

![Fig. 2.42: Flow lines around an aerofoil](image)

If the air over the top surface travels faster than the air below the wing, this decreases the air pressure above the wing. The air flowing below the wing moves almost in a straight line, so its speed and air pressure remain the same. The air under the wings therefore pushes upward more than the air on top of the wings pushes downward, thus producing an upward force \( \vec{F} \). It is the pressure difference that generates this force. The component of \( \vec{F} \) perpendicular to the direction of motion is called the aerodynamic lift or, simply, the lift. The component parallel to the direction of flight is the drag. The lift is the force that allows an aeroplane to get off the ground and stay in the air. For an aeroplane to stay in level flight, the lift is equal in magnitude and opposite in direction to the force of gravity.

[Note: For an airborne aeroplane to get to the ground, the direction of \( \vec{F} \) must be reversed. Then, the upper surface should be more concave than the lower surface such that air above the wing travels slower than the air below it, decreasing the air pressure below the wing. This is achieved by small flaps, called ailerons, attached at the trailing end of each wing.]

**Q. 124. Explain the working of an atomizer.**  

**OR**

A perfume bottle or atomizer sprays a fluid that is in the bottle. How does the fluid rise up in the vertical tube in the bottle?  
(2 marks)

**Ans.** An atomizer is a device which entrap or entrains liquid droplets in a flowing gas. Its working is based on Bernoulli’s principle. A squeeze bulb or a
pump is used to create a jet of air over an open tube dipped into a liquid. By Bernoulli’s principle, the high-velocity air stream creates low pressure at the open end of the tube. This causes the liquid to rise in the tube. The liquid is then dispersed into a fine spray of droplets. This type of system is used in a perfume bottle, a paint sprayer, insect and perfume sprays and an automobile carburetor.

[Notes: A Bunsen burner uses an adjustable gas nozzle to entrain air into the gas stream for proper combustion. Aspirators, used as suction pumps, in dental and surgical situations (for draining body fluids) or for draining a flooded basement, is another similar application. Some chimney pipes have a T-shape, with a cross-piece on top that helps draw up gases whenever there is even a slight breeze.]

Q. 125. Roofs are sometimes blown off vertically during a tropical cyclone, and houses sometimes explode outward when hit by a tornado. Use Bernoulli’s principle to explain these phenomena. (1 mark)

Ans. A cyclonic high wind blowing over a roof creates a low pressure above it, in accordance with Bernoulli’s principle. The pressure below the roof is equal to the atmospheric pressure which is now greater than the pressure above the roof. This pressure difference causes an aerodynamic lift that lifts the roof up. Once the roof is lifted up, it blows off in the direction of the wind.

Wind speeds in a tornado may be much higher and thus create much greater pressure differences. Sometimes, wooden houses hit by a tornado explode.

Q. 126. Describe what happens: (1) Hold the short edge of a paper strip (2” x 6”) up to your lips—the strip slanting downward over your finger—and blow over the top of the strip. (2) Hold two strips of paper up to your lips, separated by your fingers and blow between the strips. (I mark each)

Ans. (1) The air stream over the top surface travels faster than the air stream below the paper strip. This decreases the air pressure above the strip relative to that below. This produces an aerodynamic lift in accordance with Bernoulli’s principle and the paper strip will lift up.

(2) Air passing between the paper strips flows in a narrower channel and, in accordance with Bernoulli’s principle, must increase its speed, causing the pressure between them to drop. This will pinch the two strips together.

Learn this as well...

Why is it preferable for airplane to take off into the wind rather than with the wind?

Pilots typically take off into the wind (hence called the headwind) instead of with the wind (hence called the tailwind) because it reduces the required ground speed to attain the airspeed for “wheels up”. The speed of a headwind gets added to a plane’s ground speed. While a tailwind speed must be subtracted from the plane’s ground speed. For example, for a plane that can take off at, say 50 knots airspeed, the plane’s required ground speed only 40 knots if there is a headwind of 10 knots. On the other hand, if there is a tailwind of 10 knots, trying to reach an airspeed of 50 knots will require a faster ground speed, and hence a risk of running out of runway.

So in essence, taking off into the wind provides additional lift, helping to take off more quickly, i.e., in a shorter distance on the runway. Usually, the busiest airports are designed to allow pilots to take off into the wind. Also, pilots don’t just take off into the wind, they also land in it because it allows to land in a shorter distance on the runway.

Taking off with a tailwind is pretty much never advisable unless it’s a oneway strip or runway, or the winds are below the tailwind limitations for that aircraft type.
Q. 127. Solve the following:

1. Water is flowing through a horizontal pipe of varying cross section. At a certain point where the velocity is 0.12 m/s, the pressure of water is 0.010 m of mercury. What is the pressure at a point where the velocity is 0.24 m/s? (3 marks)

Solution:

Data: \( v_1 = 0.12 \text{ m/s}, \ p_1 = 0.010 \text{ m of Hg}, \ v_2 = 0.24 \text{ m/s}, \ y_1 = y_2 \) (horizontal pipe), \( \rho_{hw} = 13600 \text{ kg/m}^3, \ \rho_w = 1000 \text{ kg/m}^3, \ g = 9.8 \text{ m/s}^2 \)

\[ p_1 = 0.010 \text{ m of Hg} \]
\[ = (0.010 \text{ m}) (13600 \text{ kg/m}^3)(9.8 \text{ m/s}^2) \]
\[ = 133.28 \text{ Pa} \approx 133 \text{ Pa} \]

According to Bernoulli’s principle,
\[ p_1 + \frac{1}{2} \rho_w v_1^2 + \rho_w g y_1 = p_2 + \frac{1}{2} \rho_w v_2^2 + \rho_w g y_2 \]
\[ \therefore p_1 + \frac{1}{2} \rho_w v_1^2 = p_2 + \frac{1}{2} \rho_w v_2^2 \] (\( y_1 = y_2 \))
\[ \therefore p_2 = p_1 + \frac{1}{2} \rho_w (v_2^2 - v_1^2) \]
\[ = (1333 \text{ Pa}) + \frac{1}{2}(1000 \text{ kg/m}^3)[(0.12 \text{ m/s})^2 - (0.24 \text{ m/s})^2] \]
\[ = 1333 + 500 (0.12 + 0.24) (0.12 - 0.24) \]
\[ = 1333 - 500 (0.36) (0.12) \]
\[ = 1333 - 21.6 \]
\[ = 1311.4 \text{ Pa} \]
\[ \approx \frac{1311 \text{ Pa}}{(13600 \text{ kg/m}^3)(9.8 \text{ m/s}^2)} \]
\[ = 0.009836 \text{ m of Hg} \]

This gives the pressure of the water in the pipe where the flow velocity is 0.24 m/s.

2. A building receives its water supply through an underground pipe 2 cm in diameter at an absolute pressure of \( 4 \times 10^5 \text{ Pa} \) and flow velocity 4 m/s. The pipe leading to higher floors is 1.5 cm in diameter. Find the flow velocity and pressure at the floor inlet 10 m above. (4 marks)

Solution:

Data: \( d_1 = 3 \text{ cm}, \ p_1 = 4 \times 10^5 \text{ Pa}, \ v_1 = 4 \text{ m/s}, \)
\[ d_2 = 2 \text{ cm}, \ h_2 - h_1 = 10 \text{ m} \]

By continuity equation, the flow velocity at the higher floor inlet

\[ v_2 = v_1 \frac{A_1}{A_2} = v_1 \left( \frac{d_1}{d_2} \right)^2 = (4 \text{ m/s}) \left( \frac{3 \text{ cm}}{2 \text{ cm}} \right)^2 = 9 \text{ m/s} \]

By Bernoulli’s equation,
\[ p_1 + \frac{1}{2} \rho v_1^2 + \rho g h_1 = p_2 + \frac{1}{2} \rho v_2^2 + \rho g h_2 \]
\[ \therefore p_2 = p_1 - \frac{1}{2} \rho (v_2^2 - v_1^2) - \rho g (h_2 - h_1) \]
\[ = (4 \times 10^5) - \frac{1}{2}(10^3)(9^2 - 4^2) - (10^3)(10) \]
\[ = (400 - 32.5 - 100) \times 10^3 \]
\[ = 2.675 \times 10^5 \text{ Pa} = 267.6 \text{ kPa} \]

3. Calculate the total energy per unit mass possessed by water at a point where the pressure is \( 0.1 \times 10^5 \text{ N/m}^2 \), the velocity is 0.02 m/s and the height of the water level from the ground is 10 cm. Density of water \( = 1000 \text{ kg/m}^3 \). (3 marks)

Solution:

Data: \( p = 0.1 \times 10^5 \text{ N/m}^2 = 10^4 \text{ Pa}, \ v = 0.02 \text{ m/s}, \ y = 10 \text{ cm} = 0.1 \text{ m}, \ \rho = 1000 \text{ kg/m}^3, \ g = 9.8 \text{ m/s}^2 \)

The total energy per unit mass of water
\[ \frac{p}{\rho} + \frac{1}{2} v^2 + gy \]
\[ = \frac{10^4 \text{ Pa}}{10^3 \text{ kg/m}^3} + \frac{1}{2}(2 \times 10^{-2} \text{ m/s})^2 \]
\[ + (9.8 \text{ m/s}^2)(0.1 \text{ m}) \]
\[ = 10 + 0.0002 + 0.98 = 10.9802 \text{ J/kg} \]

4. A horizontal wind with a speed of 11 m/s blows past a tall building which has large windowpanes of plate glass of dimensions 4 m × 1.5 m. The air inside the building is at atmospheric pressure. What is the total force exerted by the wind on a window pane? [Density of air \( = 1.3 \text{ kg/m}^3 \)] (3 marks)

Solution:

Data: \( v_1 \) (inside) = 0 m/s, \( v_2 \) (outside) = 11 m/s, \( \rho = 1.3 \text{ kg/m}^3, \ p_1 = p_0 = \text{ atmospheric pressure}, \ A = 4 \text{ m} \times 1.5 \text{ m} = 6 \text{ m} \)

Let \( p_2 \) be the air pressure outside a window. At the same height, Bernoulli’s equation gives
\[ p_1 + \frac{1}{2} \rho v_1^2 = p_2 + \frac{1}{2} \rho v_2^2 \]
\[ \therefore p_0 = p_2 + \frac{1}{2} \rho v_2^2 \]
\[ \therefore \text{The difference in pressure across a window pane is} \]
(5) A water tank has a hole at a distance \( x \) from the free surface of water in the tank. If the radius of the hole is 2 mm and the velocity of efflux is 11 m/s, find \( x \). (2 marks)

**Solution:**

Data: \( r = 2 \) mm, \( v = 11 \) m/s, \( g = 9.8 \) m/s\(^2\)

By Torricelli’s law of efflux, the velocity of efflux,

\[
\frac{v^2}{2g} = \left(\frac{11 \text{ m/s}}{2(9.8 \text{ m/s}^2)}\right)
\]

\[
\frac{121}{19.6} = 6.173 \text{ m}
\]

(6) The pressure of water inside a closed pipe is \( 3 \times 10^5 \) N/m\(^2\). This pressure reduces to \( 2 \times 10^5 \) N/m\(^2\) on opening the valve of the pipe. Calculate the speed of water flowing through the pipe. [Density of water = 1000 kg/m\(^3\)] (2 marks)

**Solution:**

Data: \( p_1 = 3 \times 10^5 \) Pa, \( v_1 = 0 \), \( p_2 = 2 \times 10^5 \) Pa, \( \rho = 10^3 \) kg/m\(^3\)

Assuming the potential head to be zero, i.e., the pipe to be horizontal, the Bernoulli equation is

\[
p_1 + \frac{1}{2} \rho v_1^2 = p_2 + \frac{1}{2} \rho v_2^2
\]

\[
\therefore \frac{v_2^2}{2} = \frac{2(p_1 - p_2)}{\rho} \quad \text{[}\therefore v_1 = 0\text{]}^2
\]

\[
= \frac{2(3 - 2) \times 10^5}{10^3} = 200
\]

\[
\therefore \frac{v_2}{\sqrt{2}} = 14.14 \text{ m/s}
\]

(7) With what velocity does water start flowing out of an orifice in a tank with initial gauge pressure \( 4 \times 10^5 \) N/m\(^2\)? [Density of water = 1000 kg/m\(^3\)] (3 marks)

**Solution:**

Data: \( p - p_0 = 4 \times 10^5 \) Pa, \( \rho = 10^3 \) kg/m\(^3\)

If the orifice is at a depth \( h \) from the water surface in a tank, the gauge pressure there is

\[
p - p_0 = \rho gh \quad \ldots (1)
\]

By Toricelli’s law of efflux, the velocity of efflux,

\[
v = \sqrt{2gh} \quad \ldots (2)
\]

Substituting for \( h \) from Eq. (1),

\[
v = \sqrt{2g \frac{p - p_0}{\rho}} = \sqrt{\frac{2(p - p_0)}{\rho}}
\]

\[
= \sqrt{\frac{2(4 \times 10^5)}{10^3}} = 20 \sqrt{2} = 28.28 \text{ m/s}
\]

**Multiple Choice Questions**

Q. 128. Choose the correct option: (1 mark each)

1. A hydraulic lift is designed to lift heavy objects of maximum mass 2000 kg. The area of cross section of piston carrying the load is \( 2.25 \times 10^{-2} \) m\(^2\). What is the maximum pressure the smaller piston would have to bear?

   (A) \( 0.8711 \times 10^6 \) N/m\(^2\)  
   (B) \( 0.5862 \times 10^7 \) N/m\(^2\)  
   (C) \( 0.4869 \times 10^5 \) N/m\(^2\)  
   (D) \( 0.3271 \times 10^4 \) N/m\(^2\)

2. Two capillary tubes of radii 0.3 cm and 0.6 cm are dipped in the same liquid. The ratio of heights through which the liquid will rise in the tubes is

   (A) 1 : 2  
   (B) 2 : 1  
   (C) 1 : 4  
   (D) 4 : 1.

3. The energy stored in a soap bubble of diameter 6 cm and surface tension 0.04 N/m, is nearly

   (A) \( 0.9 \times 10^{-5} \) J  
   (B) \( 0.4 \times 10^{-5} \) J  
   (C) \( 0.7 \times 10^{-5} \) J  
   (D) \( 0.5 \times 10^{-3} \) J.
4. Two hailstones with radii in the ratio of 1 : 4 fall from a great height through the atmosphere. Then the ratio of their terminal velocities is
(A) 1 : 2   (B) 1 : 12   (C) 1 : 16   (D) 1 : 8.

5. In Bernoulli’s theorem, which of the following is conserved?
(A) linear momentum   (B) angular momentum
(C) mass   (D) energy

6. Consider the following statements:
I. A fluid in hydrostatic equilibrium exerts only normal force on any surface within the fluid.
II. A fluid can resist a tangential force.
Of these,
(A) only (I) is correct   (B) only (II) is correct
(C) both are correct   (D) both are false.

7. Which of the following is correct?
(A) The free surface of a liquid at rest is horizontal.
(B) The pressure at a point within a liquid at rest is the same in all directions.
(C) The pressure at all points within a liquid at rest is the same.
(D) Both (A) and (B).

8. The surface of a liquid (of uniform density \( \rho \)) in a container is open to the atmosphere. The atmospheric pressure is \( p_0 \). The pressure \( \rho gh \), at a depth \( h \) below the surface of the liquid, is called the
(A) absolute pressure   (B) normal pressure
(C) gauge pressure   (D) none of these.

9. Three vessels having the same base area are filled with water to the same height, as shown. The force exerted by water on the base is
(A) largest for vessel P   (B) largest for vessel Q
(C) largest for vessel R   (D) the same in all three.

10. Rain drops or liquid drops are spherical in shape, especially when small, because
(A) cohesive force between the molecules of water have spheres of influence
(B) for a given volume, a spherical drop has the least surface energy
(C) for a given volume, a spherical drop has the maximum surface energy
(D) the pressure inside a drop is many times the atmospheric pressure outside.

11. The surface tension acts
(A) perpendicular to the surface and vertically upwards
(B) perpendicular to the surface and vertically into the liquid
(C) tangential to the surface
(D) only at the liquid-solid interface.

12. A thin ring of diameter 8 cm is pulled out of water (surface tension 0.07 N/m). The force required to break free the ring from water is
(A) 0.0088 N   (B) 0.0176 N
(C) 0.0352 N   (D) 3.52 N.

13. A matchstick 5 cm long floats on water. The water film has a surface tension of 70 dyn/cm. A little comphor put on one side of stick reduces the surface tension there to 50 dyn/cm. The net force on the matchstick is
(A) 4 dynes   (B) 20 dynes
(C) 100 dynes   (D) 600 dynes.

14. A big drop of radius \( R \) is formed from 1000 droplets of water. The radius of a droplet will be (Oct. ’13)
(A) \( 10 R \)   (B) \( R \frac{1}{10} \)
(C) \( R \frac{1}{100} \)   (D) \( R \frac{1}{1000} \).

15. The work done in breaking a spherical drop of a liquid (surface tension \( T \)) of radius \( R \) into 8 equal drops is
(A) \( \pi R^2 T \)   (B) \( 2\pi R^2 T \)
(C) \( 3\pi R^2 T \)   (D) \( 4\pi R^2 T \).

16. If for a liquid in a vessel, the force of cohesion is more than the force of adhesion,
(A) the liquid does not wet the solid
(B) the liquid wets the solid
(C) the surface of the liquid is plane
(D) the angle of contact is zero.

17. If a liquid does not wet a solid surface, its angle of contact with the solid surface is
(A) zero   (B) acute   (C) 90°   (D) obtuse.
18. The pressure within a bubble is higher than that outside by an amount proportional
   (A) directly to both the surface tension and the bubble size
   (B) directly to the surface tension and inversely to the bubble size
   (C) directly to the bubble size and inversely to the surface tension
   (D) inversely to both the surface tension and the bubble size.

19. The pressure difference across the surface of a spherical water drop of radius 1 mm and surface tension 0.07 N/m is
   (A) 28 Pa (B) 35 Pa (C) 140 Pa (D) 280 Pa.

20. An air bubble just inside a soap solution and a soap bubble blown using the same solution have their radii in the ratio 3 : 2. The ratio of the excess pressure inside them is
   (A) 1 : 12 (B) 1 : 6 (C) 1 : 3 (D) 1 : 2.

21. A liquid rises to a height of 5 cm in a glass capillary tube of radius 0.02 cm. The height to which the liquid would rise in a glass capillary tube of radius 0.04 cm is
   (A) 2.5 cm (B) 5 cm (C) 7.5 cm (D) 10 cm.

22. In a gravity free space, the liquid in a capillary tube will rise to
   (A) the same height as that on the Earth
   (B) a lesser height than on the Earth
   (C) slightly more height than that on the Earth
   (D) the top and overflow.
   [see the note in the answer.]

23. In which of the following substances, does the surface tension increase with an increase in temperature?
   (A) Copper (B) Molten copper (C) Iron (D) Molten iron

24. A fluid flows in steady flow through a pipe. The pipe has a circular cross section, but its radius varies along its length. The mass of the fluid passing per second at the entrance point (radius $R$) of the pipe is $Q_1$, while that at the exit point (radius $R/2$) is $Q_2$. Then, $Q_2$ is equal to
   (A) $\frac{1}{4} Q_1$ (B) $Q_1$ (C) $2Q_1$ (D) $4Q_1$.

25. The dimensions of coefficient of viscosity are
   (A) $[ML^{-1}T^{-2}]$ (B) $[ML^{-1}T^{-2}]$ (C) $[ML^{-2}]$ (D) $[ML^{-1}T^{-1}]$.

26. The unit of coefficient of viscosity is
   (A) the pascal-second (B) the pascal (C) the poise-second (D) both (A) and (C).

27. A fluid flows past a sphere in streamline flow. The viscous force on the sphere is directly proportional to
   (A) the radius of the sphere (B) the speed of the flow (C) the coefficient of viscosity of the fluid (D) all of these.

28. Water flows in a streamlined flow through the pipe shown in the following figure. The pressure
   (A) is greater at A than at B (B) at A equals that at B (C) is less at A than at B (D) at A is unrelated to that at B.

29. Two steel marbles (of radii $R$ and $R/3$) are released in a highly viscous liquid. The ratio of the terminal velocity of the larger marble to that of the smaller is
   (A) 9 (B) 3 (C) 1 (D) $\frac{1}{9}$.

30. A large tank, filled with a liquid, is open to the atmosphere. If the tank discharges through a small hole at its bottom, the speed of efflux does NOT depend on
   (A) cross-sectional area of the hole (B) depth of the hole from the liquid surface (C) acceleration due to gravity (D) all of these.

Ans. 1. (A) $0.8711 \times 10^6$ N/m$^2$ 2. (B) 2 : 1 3. (A) $0.9 \times 10^{-3}$ J 4. (C) 1 : 16 5. (D) energy 6. (A) only (I) is correct 7. (D) Both (A) and (B) 8. (C) gauge pressure 9. (D) the same in all three 10. (B) for a given volume, a spherical drop
has the least surface energy

11. (C) tangential to the surface.

12. (C) 0.0352 N

13. (C) 100 dynes

14. (B) \( \frac{R}{10} \)

15. (D) \( 4\pi R^2 T \)

16. (A) the liquid does not wet the solid

17. (D) obtuse

18. (B) directly to the surface tension and inversely to the bubble size

19. (C) 140 Pa

20. (C) 1:3

21. (A) 2.5 cm

22. (D) the top and overflow

23. (B) Molten copper

24. (B) \( Q \)

25. (D) \([\text{ML}^{-1}\text{T}^{-1}]\)

26. (A) the pascal-second

27. (D) all of these

28. (A) is greater at A than at B

29. (A) 9

30. (A) cross-sectional area of the hole.

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**Formulae At a Glance**

1. Gauge pressure due to a fluid column = \( h \rho g \). Absolute pressure = \( p_0 + h \rho g \).

2. Pascal’s law : \( \frac{F_1}{A_1} = \frac{F_2}{A_2} \)

3. \( T = \frac{F}{l} = \frac{W}{dA} \)

4. Work done in blowing a soap bubble by increasing its radius from \( r_1 \) to \( r_2 \), \( W = 8\pi T (r_2^2 - r_1^2) \)

5. When \( n \) drops (each of radius \( r \)) coalesce into a single drop of radius \( R \), or a single drop of radius \( R \) breaks into \( n \) drops (each of radius \( r \)),

\[ \therefore R^3 = nr^3 \]

Energy released/absorbed = \( 4\pi T (nr^2 - R^2) \)

6. For a cavity (gas bubble inside a liquid), \( p - p_0 = \frac{2T}{R} \)

7. For a soap bubble, \( p - p_0 = \frac{4T}{R} \)

8. \( h = \frac{2T \cos \theta}{\rho g} \), \((h \gg r)\)

9. \( T = T_0 (1 - \alpha \theta) \)

10. \( Re = \frac{\nu_d \rho}{\eta} \)

11. \( \frac{F}{A} = \eta \frac{dv}{dy} \), \( \frac{F}{A} = \eta \frac{dv}{dy} \)

12. \( |f| = 6\pi \eta rv_0 \)

13. \( v_1 = \frac{2r^2 (\rho - \rho_L) g}{9\eta} \)

14. Equation of continuity : \( A_1 \rho_1 v_1 = A_2 \rho_2 v_2 \). For an incompressible fluid, \( A_1 v_1 = A_2 v_2 \)

15. \( p + \frac{1}{2} \rho v^2 + \rho g y = \text{constant} \) or \( \frac{p}{\rho} + \frac{1}{2} v^2 + g y = \text{constant} \)

16. Venturi meter : \( v_{\text{inlet}} = \sqrt{\frac{2\rho_{\text{inlet}} gh}{\rho \left[ \left( \frac{A_1}{A_2} \right)^2 - 1 \right]}} \)

OR \( v_{\text{inlet}} = \sqrt{\frac{2gh}{\left( \frac{A_1}{A_2} \right)^2 - 1}} \)

17. Velocity of efflux, \( v = \sqrt{2 \left( \frac{p - p_0}{\rho} + gh \right)} \) For a tank open to atmosphere, \( v = \sqrt{2gh} \)

(3 marks) (Ans. 19.79 m/s)
2.1 – 2.3

1. What is the absolute pressure 15 m below the surface of a fresh-water reservoir? (2 marks) (Ans. 2.483 \times 10^5 \text{ Pa})

2. A tank is filled with water up to a height of 4 m. Find the pressure exerted by the water on the bottom of the tank. Also find the absolute pressure on the bottom. (Atmospheric pressure = 1.013 \times 10^5 \text{ N/m}^2] (2 marks) (Ans. 3.92 \times 10^4 \text{ Pa}, 1.405 \times 10^5 \text{ Pa})

3. Two different liquids of density \( \rho_1 \) and \( \rho_2 \) exert the same pressure at a certain point. What is the ratio of the heights of the respective liquid columns? (3 marks) (Ans. \( h_1/h_2 = \rho_2/\rho_1 \))

4. A swimmer is swimming in a swimming pool at 6 m below the surface of the water. Calculate the pressure on the swimmer due to the water above. (Density of water = 1000 kg/m\(^3\)) (1 mark) (Ans. 5.88 \times 10^5 \text{ Pa})

5. Water flows through a pipe of diameter 2 cm at a speed of 1 m/s. What should be the diameter of a nozzle if the water is to emerge at a speed of 20 m/s? (2 marks) (Ans. 0.4472 cm)

6. A hydraulic car lift is used to lift a car of mass 3000 kg. The piston on which the car is supported has a cross section of 425 cm\(^2\). Find the pressure that has to be exerted by the smaller piston if both the pistons are at the same horizontal level. (2 marks) (Ans. 6.92 \times 10^5 \text{ Pa})

7. The pistons of a hydraulic lift have cross-sectional areas of 3 cm\(^2\) and 200 cm\(^2\). What force must be applied to the small piston for the lift to raise a load of 20 kN? (2 marks) (Ans. 300 N)

8. The radii of two pistons in a hydraulic press are 4 cm and 16 cm respectively. A force of 25 N is applied to the smaller piston. Find the force exerted on the larger piston. (2 marks) (Ans. 400 N)

9. A hydraulic brake system of a car of mass 1000 kg having speed of 50 km/h, has a cylindrical piston of radius of 0.5 cm. The slave cylinder has a radius of 2.5 cm. If a constant force of 100 N is applied on the brake what distance the car will travel before coming to stop? (3 marks) (Ans. 38.58 m)

10. A beaker of radius 10 cm is filled with water. Calculate the force of surface tension on any diametrical line on its surface. Surface tension of water is 0.075 N/m. (3 marks) (Ans. 15 mN)

11. Calculate the work done in blowing a soap bubble to a radius of 1 cm. The surface tension of soap solution is 2.5 \times 10^{-2} \text{ N/m}. (Ans. 0.075 \mu J)

12. Calculate the work done in blowing a soap bubble of radius 0.05 m. The surface tension of soap solution is 27 dyn/cm. (2 marks) (Ans. 1.696 \times 10^{-3} \text{ J})

13. A soap bubble in air expands so that its diameter increases from 2 cm to 5 cm. Calculate the work done if the surface tension of soap solution is 27 \times 10^{-3} \text{ N/m}. (2 marks) (Ans. 1.424 \times 10^{-3} \text{ J})

14. A spherical drop of mercury, of radius 1 mm, falls from a height on a flat surface and breaks into one thousand spherical droplets, all of equal size. Assuming no loss of energy, determine the height from which the drop fell. [Surface tension of mercury = 0.466 N/m, density of mercury = 1.36 \times 10^4 \text{ kg/m}^3] (2 marks) (Ans. 9.441 \times 10^{-2} \text{ m})

15. Two mercury drops of radii 2 mm and 3 mm coalesce to form a single drop. What is the change in the surface energy of the drops? [Surface tension of mercury = 0.466 N/m] (2 marks) (Ans. 9.265 \times 10^{-4} \text{ J})

16. 64 water droplets combine to form a single drop. What is the ratio of the total surface energy of all the droplets to that of the larger drop? \([T = 0.072 \text{ N/m}]\) (2 marks) (Ans. 4)

17. A soap film is formed between two straight parallel wires, 5 mm apart and each 10 cm long. Calculate the work required to stretch the film by 1 mm. \([T = 0.035 \text{ N/m}]\) (2 marks) (Ans. 7.0 \mu J)

18. What should be the diameter of a water drop so that the excess pressure inside it is 80 N/m\(^2\)? \([T_{\text{water}} = 7.27 \times 10^{-2} \text{ N/m}]\) (2 marks) (Ans. 3.6 mm)
19. What is the pressure inside a small air bubble of diameter 0.2 mm just below the free surface of water? 
\[ T = 0.072 \text{ N/m}, \text{atmospheric pressure} = 101.3 \text{ kPa} \]
(2 marks) (Ans. 102.74 kPa)

20. The excess pressure inside a spherical soap bubble of radius 0.01 m is balanced by that due to column of olive oil of density 0.92 g/cm³ and height 1.81 mm. Calculate the surface tension of the soap solution.
(2 marks) (Ans. 0.0408 N/m)

21. The lower end of a capillary tube of diameter 2.0 mm is dipped 8 cm below the surface of water in a beaker. What is (i) the pressure required in the tube in order to blow a hemispherical bubble at its end inside water (ii) the excess pressure inside the bubble?

Surface tension of water = 0.073 N/m, 1 atm = 101.3 kPa, density of water = 1000 kg/m³, \( g = 9.8 \text{ m/s}^2 \)
(3 marks) (Ans. 1.022 \times 10^5 \text{ Pa}, 146 \text{ Pa})

22. A glass capillary of radius 1 mm is partially immersed into a liquid of density 800 kg/m³. If the surface tension of the liquid is 5 \times 10^{-2} \text{ N/m} and its angle of contact with glass is 30°, calculate the height to which the liquid rises in the capillary tube.
(2 marks) (Ans. 0.011 m or 1.1 cm)

23. A liquid of density 900 kg/m³ rises to a height of 1.8 cm in a capillary tube of diameter 1.2 mm. If the angle of contact is 25°, find the surface tension of the liquid. (2 marks) (Ans. 5.255 \times 10^{-2} \text{ N/m})

24. Water rises to a height of 10 cm in a certain capillary. If the same capillary is dipped into mercury, the level of mercury decreases by 4.09 cm. Compare the surface tension of mercury to that of water.
\[ \rho_{\text{mercury}} = 13.6 \times 10^3 \text{ kg/m}^3; \rho_{\text{water}} = 10^3 \text{ kg/m}^3. \]
The angle of contact for water is 0° and that for mercury is 148°.]
(2 marks) (Ans. \( T_{\text{mercury}}/T_{\text{water}} = 6.559 \))

25. A capillary tube of radius 5 \times 10^{-4} \text{ m} is immersed in a beaker filled with mercury. The mercury meniscus inside the capillary tube is 8 \times 10^{-3} \text{ m} below the mercury level in the beaker. Determine the angle of contact between mercury and glass.

Surface tension of mercury is 0.465 \text{ N/m} and its density is 13.6 \times 10^3 \text{ kg/m}³
(2 marks) (Ans. 124°58’)

26. A U-tube containing water is held vertical. If the diameters of its arms are 5 mm and 1 cm, find the difference between the levels of water in the two arms. The surface tension, angle of contact with glass and density of water are respectively 0.07 N/m, 0° and 1000 kg/m³. (2 marks) (Ans. 5.71 mm)

27. A glass capillary tube of diameter 0.15 mm is dipped in glycerine whose surface tension is 0.063 N/m. Determine the angle of contact if glycerine rises to a height of 0.1361 m in the tube. [Density of glycerine = 1260 kg/m³] (2 marks) (Ans. 0°)

28. A glass rod of diameter 2 mm is inserted symmetrically inside a vertical capillary tube of inner diameter 3.6 mm. If the lower end of the arrangement is dipped in water, to what height will water rise inside? [Surface tension of water = 0.072 N/m, angle of contact = 0°]

[Hint : The net upward force due to surface tension = \( 2\pi r (r_{0} + r_{i}) \cos \theta \), the weight of the water column = \( \rho [\pi (r_{0}^2 - r_{i}^2)h]g \).]
(2 marks) (Ans. 1.837 cm)

29. A capillary tube with an inner diameter of 0.25 mm can support a 10 cm column of a liquid that has a density of 930 kg/m³. The observed contact angle is 15°. What is the surface tension of the liquid?
(2 marks) (Ans. 5.897 \times 10^{-2} \text{ N/m})

30. A mercury barometer is to be made from a glass tube with an inner radius R. If the capillarity correction on the barometric height is to be less than 0.5%, what should be the minimum value of R? [For mercury, surface tension = 0.472 N/m, contact angle with glass = 148°, density = 13.6 \times 10^3 \text{ kg/m}³, barometric height = 0.76 m]

[Hint : \(|h| < 0.005 \times 0.76 \text{ m}, R > \left| \frac{2T \cos \theta}{\rho gh} \right| \)]
(2 marks) (Ans. 1.58 mm)

26–2.7

31. Calculate the viscous force acting on a layer of water of surface area 100 cm², if the relative velocity between two layers separated by 0.5 mm is 6 cm/s. The coefficient of viscosity of water is 10^{-3} \text{ Pa s}.
(2 marks) (Ans. 0.12 N)
32. Calculate the force due to viscosity acting on a layer of water of surface area \(2 \times 10^{-2} \text{ m}^2\), if the relative velocity between two layers separated by 0.4 mm is 5 cm/s. [Coefficient of viscosity of water = 0.01 poise] (2 marks) (Ans. \(2.5 \times 10^{-3} \text{ N}\))

33. A metal plate with an area of 50 cm\(^2\) is in contact with a layer of liquid with a thickness 1 mm. If the coefficient of viscosity of the liquid is 2 Ns/m\(^2\), find the horizontal force needed to move the plate along the surface of the liquid with a velocity of 15 cm/s. (2 marks) (Ans. 1.5 N)

34. A metal plate of length 20 cm and breadth 2 cm is in contact with a layer of oil 0.8 mm thick. The horizontal force required to move it with a velocity of 6 cm/s along the surface of the oil is 0.54 N. Find the coefficient of viscosity of the oil. (2 marks) (Ans. 1.8 Ns/m\(^2\))

35. Calculate the viscous force acting on a rain drop of diameter 4 mm, falling with a constant velocity of 4 m/s through air. The coefficient of viscosity of air is \(1.8 \times 10^{-4} \text{ poise}\). [1 P = 0.1 Pa\(\cdot\)s] (2 marks) (Ans. \(2.71 \times 10^{-6} \text{ N}\))

36. A sphere of radius 3 mm falls through a column of glycerine at 20 °C with a terminal speed of 10 cm/s. Find the force due to viscosity acting on the sphere. [\(\eta\) of glycerine at 20 °C = 1.34 Ns/m\(^2\)] (2 marks) (Ans. \(7.573 \times 10^{-3} \text{ N}\))

37. A sphere of radius 5 mm falls through a liquid with a terminal speed of 20 cm/s. Find the coefficient of viscosity of the liquid, if the force due to viscosity acting on the sphere is \(2 \times 10^3 \text{ dynes}\). (2 marks) (Ans. 10.61 poise)

38. A steel ball of radius 0.3 mm falls through a tube of glycerin with a velocity of 2 m/s at time \(t\). Coefficient of viscosity of glycerine = 0.833 Ns/m\(^2\). Determine the viscous force acting on the steel ball at that time. (3 marks) (Ans. \(9.422 \times 10^{-3} \text{ N}\))

39. A spherical drop of oil falls at a constant speed of 4 cm/s through still air. Calculate the radius of the drop. [Density of the oil = 0.9 g/cm\(^3\), density of air = 1.0 g/cm\(^3\), coefficient of viscosity of air = \(1.8 \times 10^{-4} \text{ poise}\), \(g = 980 \text{ cm/s}^2\)]. (3 marks) (Ans. 0.574 cm)

40. A metal sphere of radius 1 mm falling under gravity through a viscous liquid acquires a terminal velocity of 4 cm/s. If the density of the metal is 8000 kg/m\(^3\) and that of the liquid is 1200 kg/m\(^3\), calculate the coefficient of viscosity of the liquid. (2 marks) (Ans. 0.37 Pa\(\cdot\)s)

41. A sphere of radius 3 mm falls through a liquid of density 1.2 g/cm\(^3\). Find the terminal speed of the sphere, if the density of the material of the sphere is 7.2 g/cm\(^3\) and the coefficient of viscosity of the liquid is 15 poise. (2 marks) (Ans. 7.84 cm/s)

42. A sphere of radius 2 mm falls through a liquid of density 1.3 g/cm\(^3\) with a terminal speed of 5 cm/s. The density of the material of the sphere is 8.5 g/cm\(^3\). Find the coefficient of viscosity of the liquid. (2 marks) (Ans. 12.54 poise)

43. Calculate the terminal speed of an oil drop in air, if the radius of the oil drop is \(10^{-5} \text{ m}\) and the density of the oil is 900 kg/m\(^3\). [\(\eta\) (air) = \(1.8 \times 10^{-5} \text{ Ns/m}^2\), \(\sigma\) (air) = \(\rho\) (oil)] (2 marks) (Ans. 1.089 \times 10^{-2} \text{ m/s})

44. Water flows through a horizontal pipe of non-uniform cross section. The pressure is 1 cm of mercury where the flow velocity is 25 cm/s. Find the pressure at a point where the flow velocity is 75 m/s. [\(\rho_{\text{Hg}} = 13.6 \text{ g/cm}^3\), \(\rho_{\text{water}} = 1 \text{ g/cm}^3\)] (3 marks) (Ans. 0.8126 cm of mercury)

45. Water flows horizontally through a pipeline of varying cross section. If the pressure of water is 10 m of mercury at a point where the flow velocity is 40 cm/s, what is the pressure at another point where the flow velocity is 50 cm/s? [\(\rho_{\text{Hg}} = 13.6 \text{ g/cm}^3\), \(\rho_{\text{water}} = 1 \text{ g/cm}^3\)] (3 marks) (Ans. 9.966 cm of Hg)

46. A Venturi meter has inlet diameter of 10 cm and throat diameter of 5 cm. Find the flow velocity and rate of flow of water if the pressure difference (gauge pressure) between the inlet and throat is 15 cm of mercury. [\(\rho\) (water) = \(10^3 \text{ kg/m}^3\), \(\rho_{\text{Hg}} = 13.6 \times 10^3 \text{ kg/m}^3\)]

\[
\text{Hint: } v_{\text{inlet}} = \frac{2(\rho_{\text{Hg}}/\rho)gh}{(A_t/A_i)^2 - 1}
\]

(3 marks) (Ans. 3.651 m/s, 0.02868 m\(^3\)/s or 28.68 kg/s)
47. As shown in the given figure, a piston of cross-sectional area 2 cm\(^2\) pushes a liquid out of a tube whose cross-sectional area at the outlet is 40 mm\(^2\). The piston is pushed at a rate of 2 cm/s. Determine speed at which the fluid leaves the tube.

\[ \text{(3 marks) (Ans. 0.1 m/s)} \]

48. The given figure shows the streamline flow of a non-viscous liquid of density 1000 kg/m\(^3\). The cross-sectional area at point A is 2 cm\(^2\) and that at point B is 1 cm\(^2\). The speed of the liquid at A is 5 cm/s. If both A and B are at the same horizontal level, calculate the pressure difference between A and B.

\[ \text{(3 marks) (Ans. 50 Mpa)} \]

49. Doors of a dam are 20 m below the surface of water in the dam. If one door is opened, what will be the speed of the water that flows out of the door?

\[ \text{(3 marks) (Ans. 19.79 m/s)} \]

50. Water flows through a tube as shown in the given figure. Find the difference in mercury level, if the speed of flow of water at point A is 2 m/s and at point B is 5 m/s.

\[ \text{(3 marks) (Ans. 1.07 m)} \]
2. MECHANICAL PROPERTIES OF FLUIDS

1. Pressure in a fluid
   - Effect of gravity
   - Pascal’s law
   - Hydrostatic paradox

2. Surface tension
   - Intermolecular forces
     - Cohesive
     - Adhesive
     - Range of molecular attraction
     - Sphere of influence
   - Molecular theory
     - Surface energy
     - Units & dimensions
     - Relation with surface tension
     - Angle of contact (wettability)
     - Units & dimensions (shape of meniscus or drop)
     - Laplace’s law for excess pressure

3. Fluid friction (viscosity)
   - Capillarity
     - Capillary rise (wet liquid)
     - Capillary depression (nonwet liquid)
   - Applications
     - Capillary of reduced surface tension
     - Mosquito control
     - Fabric dying
     - Blending of oils
     - Cleaning solutions (detergents, toothpaste, shampoo)
     - Medicine (antiseptic liquids)
     - Types of flow
       - Reynolds’ number

- Streamline flow
  - Velocity gradient
  - Newton’s law of viscosity
  - Stokes’ law
  - Equation of continuity
  - Bernoulli’s principle
  - Torricelli’s theorem (speed of efflux)

- Turbulent flow
  - Advantages and disadvantages
  - Applications
  - Equation of fluid flow
  - Venturi meter
  - Aerofoil
1. opentextbc.ca/physicstestbook2/chapter/variation-of-pressure-with-depth-in-a-fluid/
2. www.biolinscientific.com/measurements/surface-tension
4. en.wikipedia.org/wiki/Drop_(liquid)
5. en.wikipedia.org/wiki/Young-Laplace_equation
7. Contact angles, wetting, capillarity :
   - books.google.co.in/books?isbn=1402023022